Assignment Nr. 5

due 19 January

Problem 1

Assume that $u(t)$ is a complex and periodic function with period T that can represented by a Fourier series of the form

$$u(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t} \ , \quad \omega_n = \frac{2\pi n}{T} \quad . \tag{1}$$

Then, the Fourier coefficients, c_k, corresponding to the Fourier frequencies, ω_k, are determined by

$$c_k = \langle u(t) e^{-i\omega_k t} \rangle_T \ , \tag{2}$$

where

$$\langle \cdots \rangle_T = \frac{1}{T} \int_{-T/2}^{T/2} (\cdots) \mathrm{d}t \tag{3}$$

denotes the average over one period.

(a) Show that $\omega_{-n} = -\omega_n$.

(b) Show that

$$\langle e^{i\omega_n t} e^{-i\omega_m t} \rangle_T = \delta_{nm} \ , \tag{4}$$

where $\delta_{mn} = 1$ for $m = n$, and $\delta_{mn} = 0$ otherwise. This illustrates the *orthogonality property* of the Fourier modes.

(c) Use the orthogonality property, (4), to verify the formula for the Fourier coefficients, (2). Hint: insert (1) into (2), and assume that infinite summation and integration are interchangeable.

(d) Show that for any *real* function, $u(t)$, the Fourier coefficients exhibit *conjugate symmetry*, $c_m^* = c_{-m}$, where the asterix denotes the conjugate complex. Hint: take the conjugate complex of (2).

(e) Show that for any symmetric (but not necessarily real) function, $u(t) = u(-t)$, also the Fourier coefficients are symmetric, $c_m = c_{-m}$.

(f) Argue that if the function $u(t)$ is real and symmetric, also its Fourier coefficients are real and symmetric.

Problem 2

Consider two complex and periodic functions, $u(t)$ and $w(t)$, with period T, and zero mean. Using the definition of the average, (3), the correlation of $u(t)$ and $w(t)$ is defined by

$$R_{uw}(s) = \langle u(t) w(t+s) \rangle_T \ , \tag{5}$$

illustrating that the correlation is a function of the shift, s.

(a) Show that the correlation may be expressed as

$$R_{uw}(s) = \sum_{m=-\infty}^{\infty} c_{-m} \bar{c}_m e^{i\omega_m s} \ , \qquad (6)$$

where c_m and $\bar{c}_m$ are the Fourier coefficients of $u(t)$ and $w(t)$, respectively. Hint: insert Fourier expansions similar to (1) into (5), and then move the double sum in front of the integral. Make use of the orthogonality property, (4).

(b) Show that for any real $u(t) = w(t)$, expression (6) denotes the auto-correlation,

$$R_{uu}(s) = \frac{1}{2} \sum_{m=-\infty}^{\infty} S_m e^{i\omega_m s} = \sum_{m=1}^{\infty} S_m \cos(\omega_m s) \ , \qquad (7)$$

where $S_m = 2 c_m c_m^*$ has been introduced for convenience. Note that according to (1) the S_m are (twice) the Fourier coefficients of the auto-correlation.

(c) Show that that auto-correlation, (7), evaluated at $s = 0$ implies

$$R_{uu}(0) = \sum_{m=1}^{\infty} S_m \quad . \qquad (8)$$

Explain the contribution of S_m to the averaged energy, $\langle u^2 \rangle_T$.

Problem 3

Let's approximate (8) by a finite sum,

$$\langle u^2 \rangle_T \approx \sum_{m=1}^{N} S_m \ , \qquad (9)$$

such that ω_N denotes the highest Fourier frequency that can be represented.

(a) Show that (9) can be re-expressed as

$$\langle u^2 \rangle_T \approx \sum_{m=1}^{N} E_m \Delta\omega \ , \qquad (10)$$

where $\Delta\omega$ is the frequency difference of neighbouring Fourier modes, and

$$E_m = \frac{T}{2\pi} S_m \qquad (11)$$

is the *power spectral density*. Sometimes, it is also called the *frequency spectrum*.

(b) Comment on the similarity of (10) and the non-periodic, continuous form,

$$\langle u^2 \rangle = \int_0^{\infty} E(\omega) \mathrm{d}\omega \ , \qquad (12)$$

discussed during the lecture.

(c) Assuming that u is a velocity, what are the units of S_m and E_m?

In MATLAB, there exists an efficient function to compute the power spectral density, E_m, defined in (11). The function is called `pwelch`, and will be used in the next problem. Have a look at the manual page to understand what it does.

Problem 4

This problem uses again the time series of the three velocity components from a turbulent bottom boundary layer in the Western Baltic Sea (see Assignment 3, Problem 1). Again, you need to rotate the horizontal velocity components, such that u denotes the component pointing into the direction of the mean flow, v the spanwise component, and w the vertical component. As done in Assignment 3, remove the mean from all three components and work only with the fluctuating parts in the following. A MATLAB file containing the solution to Assignment 3 is available for download. Recall that the sampling frequency was $f = 25$Hz.

(a) Compute E_m using the function `pwelch` from the N data points representing u, v, and w. Use this function with the standard Hamming window, and approximately $N/10$ data points for the window length, overlap, and FFT length. Plot all spectra in one double-logarithmic (!) plot as a function of ω_m.

(b) At low frequencies, which velocity component contains the least energy? Can you explain this? Hint: recall that the measurements are taken close to the bottom.

(c) Plot the vertical velocity spectrum together with a curve with slope $\omega_m^{-5/3}$, as expected for the intertial subrange. For which frequency range do the slopes of the two spectra approximately coincide?

(d) Assume that in the inertial subrange the spectrum is represented by

$$E_m = C\epsilon^{2/3}\omega_m^{-5/3} \; , \qquad (13)$$

where C is a constant, and ϵ denotes the dissipation rate. Describe a method to compute the dissipation rate from your spectra computed above.