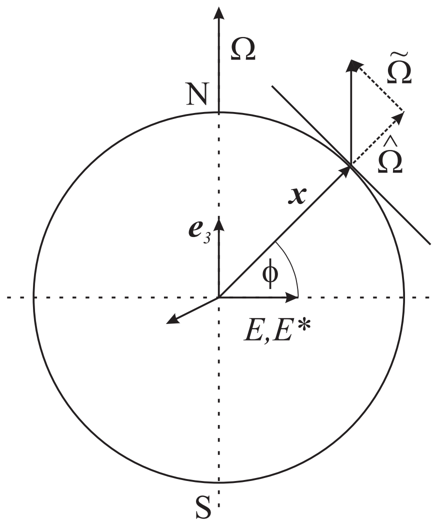
Assignment Nr. 5

due June 30th

Problem 1



Assume that the origins of a fixed reference system, E, (with fixed base vectors, $\boldsymbol{e}_i$) and a rotating reference system, E^*, (with rotating base vectors, $\boldsymbol{e}_i^*$) are located at the center of a rotating spherical planet. At the time considered, the base vectors of both coordinate systems coincide, and the base vectors $\boldsymbol{e}_3$ and $\boldsymbol{e}_3^*$ point into the direction of the planet's rotation axis indicated by $\boldsymbol{\Omega}$. The reference system E^* rotates with the planet around the same axis. The constant angular velocity of the rotation is $|\boldsymbol{\Omega}| = 2\pi/T$, where T is the time for one complete revolution of the planet.

Let $\boldsymbol{x}$ denote the position vector (with respect to the rotating reference system) of a point located at the planet's surface at the latitude ϕ. $\boldsymbol{\Omega}$ can be split into two components, one inside and one perpendicular to the planet's tangent plane at $\boldsymbol{x}$, such that $\boldsymbol{\Omega} = \tilde{\boldsymbol{\Omega}} + \hat{\boldsymbol{\Omega}}$.

(a) Argue that for the geometry discussed above, the only accelerations resulting from the motion of the reference system are the *centrifugal acceleration*, $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x})$, and the *Coriolis acceleration*, $-2\boldsymbol{\Omega} \times \boldsymbol{u}$. Why do the other acceleration terms vanish?

(b) First, consider only the Coriolis acceleration due to $\hat{\boldsymbol{\Omega}}$, perpendicular to the tangent plane. Which of the following statements are correct for a point with position vector $\boldsymbol{x}$?

(i) A point moving in the tangent plane always feels a Coriolis acceleration towards the center of the planet, except at the poles.

(ii) A point moving in the tangent plane always feels a Coriolis acceleration towards the right on the northern hemisphere and towards the left on the southern hemisphere. At the equator, it feels no Coriolis acceleration at all.

(iii) A point moving towards the center of the planet always feels a Coriolis acceleration towards the center of the planet, except at the equator.

(c) Next, consider only the Coriolis acceleration due to $\tilde{\boldsymbol{\Omega}}$, pointing towards the North. Which of the following statements are correct for the point with position vector $\boldsymbol{x}$?

(i) A point moving in the tangent plane always feels a Coriolis acceleration away from the center, except at the poles, or if it moves exactly towards the North.

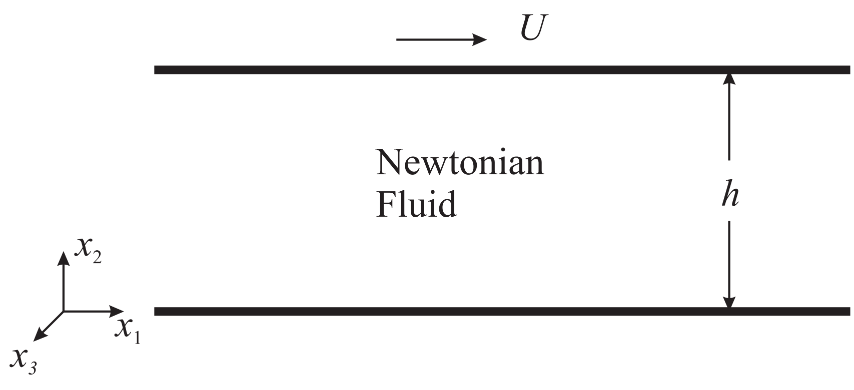
(ii) A point moving towards the center of the planet always feels a Coriolis acceleration towards the East (i.e. into the paper), except at the poles.

(iii) At the equator, a point moving towards the East always feels a Coriolis acceleration towards the North.

(d) In what directions point the components of the Coriolis acceleration resulting from $\hat{\boldsymbol{\Omega}}$ and $\tilde{\boldsymbol{\Omega}}$, respectively, for a point moving on the northern hemisphere in the tangent plane towards the East, i.e. into the paper?

(e) If an air particle is moving in the atmosphere of the planet Earth at the latitude of Rostock (ca. 54 degrees North) and moves on a windy day with a speed of 10 m s^{-1}, what are the magnitudes of the Coriolis accelerations computed in (d)? A typical value for the pressure gradient between a high and a low pressure area in the atmosphere is $1/\rho\, \partial p/\partial x = 10^{-3}$ m s^{-2}. How does this value compare to the Coriolis acceleration in the tangent plane?

Problem 2



Consider the stationary motion of an incompressible Newtonian fluid with constant density, $\rho = \rho_0$, and constant diffusivity of momentum, ν_0, between two parallel, infinite plates of distance, h. The upper plate moves parallel to the lower plate with constant velocity, U, in the $\boldsymbol{e}_1$ direction, the lower plate is at rest. The flow is homogeneous in the x_1 and x_3 directions. The only non-zero velocity component is $u_1(x_2)$. The pressure gradient is negligible.

(a) Show that for this flow the Navier-Stokes equations reduce to

$$\nu_0 \frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2^2} = 0 \quad . \tag{1}$$

Discuss, term by term, for what reason individual terms have been neglected in deriving this equation.

(b) Integrate (1) with the help of the boundary conditions $u_1(0) = 0$ and $u_1(h) = U$. Show that the velocity distribution between the plates is linear,

$$u_1 = \frac{U}{h} x_2 \ , \tag{2}$$

independent of the diffusivity. This is the velocity distribution of the Couette flow encountered already in an earlier problem.

(c) Show that, for the flow considered here, the dissipation function, $\Phi = T_{ij} S_{ij}$, is specified by

$$\frac{1}{\rho_0} \Phi = \nu_0 \frac{U^2}{h^2} \ , \tag{3}$$

independent of the spatial position.

(d) Now assume that the two bounding plates have zero heat conductivity, and that there is no external energy supply to the fluid. Argue that, under these conditions, the divergence of the heat flux vanishes, and the energy equation can be written as

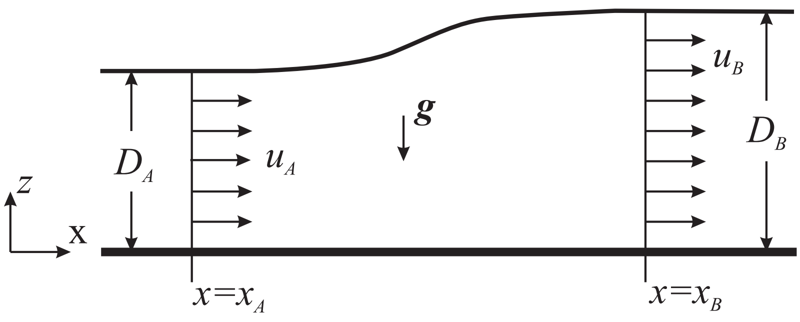
$$c_v \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{\rho_0} \Phi = \nu_0 \frac{U^2}{h^2} \ , \tag{4}$$

where c_v is heat capacity of the fluid (assumed to be constant here).

(e) Typical thermodynamical parameters for lubricating oil are $c_v = 2000$ J kg^{-1} K^{-1}, and $\nu_0 = 10^{-5}$ m^2 s^{-1}. Solve (4) for these parameters, and compute the temperature increase of the oil after $t = 60$ s for a gap of $h = 1$ mm and $U = 10$ m s^{-1}.

Problem 3

Consider the stationary flow of an inviscid fluid of constant density in a channel with flat bottom, consisting of two straight sections of different width. At $x = x_A$, located inside the first section, the width of the channel is W_A, the local water



depth is D_A, and the velocity, assumed to be vertically homogeneous, is $\boldsymbol{u}_A = u_A \boldsymbol{e}_1$. Further downstream, in the second section, at $x = x_B$, the width of the channel is W_B, the local water depth is D_B, and the velocity, again vertically homogeneous, is $\boldsymbol{u}_B = u_B \boldsymbol{e}_1$. Assume that the pressure at the free surface is equal to the constant ambient pressure, p_0.

(a) Construct a control volume including the planes $x = x_A$ and $x = x_B$, and show that the balance of mass requires that

$$u_A W_A D_A = u_B W_B D_B \quad . \tag{5}$$

(b) Assume that $z = 0$ corresponds to the flat bottom of the channel. Apply the Bernoulli equation between $x = x_A$ and $x = x_B$ along a streamline located at the free surface, and show that

$$\frac{u_A^2}{2} + g D_A = \frac{u_B^2}{2} + g D_B \ , \tag{6}$$

where g is the acceleration of gravity.

(c) Combine (5) and (6) to eliminate u_B from the problem. Show that the resulting equation is a cubic polynomial for D_B of the from

$$2gD_B^3 - \left(2gD_A + u_A^2\right) D_B^2 + \left(\frac{W_A}{W_B} D_A u_A\right)^2 = 0 \quad . \tag{7}$$

(d) Solve (7) either numerically or analytically for D_B. Plot D_B as a function of u_A for $0 \le u_A \le 2$ m s^{-1}, using the parameters $g = 9.81$ m s^{-2}, $D_A = 5$ m, and $W_A/W_B = 2$. (Hint: better don't try this without a mathematical software. Note that there a three, possibly different, roots of (7). Only one yields physically reasonable results.)