Assignment Nr. 4

due June 11th

Problem 1

Consider the one-dimensional diffusion equation,

$$\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial x^2} \, , \tag{1}$$

with constant diffusivity, ν. A general single-stage two-level scheme for (1) may be written as

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = \alpha R_j^n + \beta R_j^{n+1} \, , \tag{2}$$

where $\beta = 1 - \alpha$ for consistency. For $\alpha \neq 1$ this scheme is seen to be *implicit.* To be consistent with second order accuracy in space with the diffusion equation, the right hand side is discretized as

$$R_j^n = \nu \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{(\Delta x)^2} \, , \tag{3}$$

where Δx denotes the constant grid spacing.

(a) Show that the scheme (2) with the diffusion term discretized according to (3) can be re-expressed as

$$\begin{aligned} - &\quad \beta D \, \phi_{j-1}^{n+1} + (1 + 2\beta D) \, \phi_j^{n+1} - \beta D \, \phi_{j+1}^{n+1} \\ = &\quad \alpha D \, \phi_{j-1}^n + (1 - 2\alpha D) \, \phi_j^n + \alpha D \, \phi_{j+1}^n \, , \end{aligned} \tag{4}$$

where $D = \nu \Delta t / (\Delta x)^2$ has been defined for convenience. (5 points)

(b) By inserting into (4) an arbitrary Fourier mode,

$$\phi_j^n = a_l^n e^{i k_l j \Delta x} \, , \tag{5}$$

with wave number k_l and amplitude a_l^n, show that the amplification factor of this scheme can be written as

$$A_l = - \frac{2\alpha D (\cos k_l \Delta x - 1) + 1}{2\beta D (\cos k_l \Delta x - 1) - 1} \quad . \tag{6}$$

(10 points)

(c) To get a visual impression for the range of stability of this scheme, plot $|A_l|$ for $0 \leq k_l \Delta x \leq \pi$ using the parameters $\alpha = 1$ (explicit scheme, treated already in an earlier assignment), $\alpha = 1/2$ (Crank-Nicolson scheme), and $\alpha = 0$ (fully implicit scheme). Provide three plots of this type for $D = 0.1$, $D = 0.5$ (the stability limit of the explicit scheme), and $D = 10$, respectively. Which schemes are stable, according to your plot? (10 points)

(d) Schow that for $D \to \infty$, it follows immediately from (6) that $|A_l| \to \alpha/\beta = \alpha/(1-\alpha)$. So for which values of α one has $|A_l| \leq 1$, i.e. the scheme is stable? Schemes that are stable for arbitrarily large D are called *unconditionally stable* (5 points).

Problem 2

It has been shown during the lecture that the upstream discretization of the advection equation,

$$\phi_j^{n+1} = \phi_j^n - \mu \left(\phi_j^n - \phi_{j-1}^n \right) \ , \tag{7}$$

where μ is the Courant number, advects the l'th Fourier mode according to the discrete dispersion relation,

$$\tan \omega_l \Delta t = \frac{\mu \sin k_l \Delta x}{1 - \mu(1 - \cos k_l \Delta x)} \ , \tag{8}$$

with k_l being the wave number and ω_l the frequency of the mode; Δt and Δx are as usual the time step and the grid spacing.

(a) Show that the ratio of the physical advection speed, c, and the numerical advection speed, c_l, of the l'th Fourier mode is given by

$$\frac{c_l}{c} = \frac{1}{\mu \, k_l \Delta x} \arctan \frac{\mu \sin k_l \Delta x}{1 - \mu(1 - \cos k_l \Delta x)} \quad . \tag{9}$$

Plot this relation for the admissible range of $k_l \Delta x$ and $\mu = 0.1, 0.5, 1.0$. Is the upstream scheme accelerating, according to your plots? (10 points)

(b) Show that for very good spatial resolution, $k_l \Delta x \to 0$, the numerical speed approaches the physical speed, $c_l/c \to 1$. (Hint: consider the first terms in the Taylor-series expansions of sin, cos, and arctan.) (5 points)