Integration of TVD-advection schemes into 3D models

Hans Burchard

hans.burchard@io-warnemuende.de.

Baltic Sea Research Institute Warnemünde, Germany

Program of presentation

- TVD advection schemes: Basic ideas
- 1D test case: Linear advection
- 2D test case: solid body rotation
- Directional split technique
- Lock exchange test case
- Freshwater eddy: Role of momentum advection
- Wadden Sea Modelling: Role of momentum advection
- North Sea modelling: role of spatial resolution
- Conclusions

TVD advection schemes: Basics I

One-dimensional linear tracer equation:

$$\partial_t \Psi + \partial_x (u\Psi) = 0 \tag{1}$$

Discretisation:

$$\Psi_i^{n+1} = \Psi_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right) \tag{2}$$

Upstream-biased flux calculation:

$$F_{i+1/2}^n = \frac{1}{\Delta t} \int_{x_{i+1/2}-u\Delta t}^{x_{i+1/2}} \Psi(x')\, dx'. \tag{3}$$

TVD advection schemes: Basics II

Polynomial tracer approximation:

$$\int_{x_{i-1/2+k}}^{x_{i+1/2+k}} \tilde{\Psi}_i(x')\,dx' = \Psi_{i+k}, \quad k=-1,0,1, \tag{4}$$

Lax-Wendroff form:

$$F_{i+1/2} = \begin{cases} u_{i+1/2}\left(\Psi_i + \frac{1}{2}\Phi^+_{i+1/2}(1-|c_{i+1/2}|)(\Psi_{i+1}-\Psi_i)\right) & \text{for } u_{i+1/2} > 0, \\ u_{i+1/2}\left(\Psi_{i+1} + \frac{1}{2}\Phi^-_{i+1/2}(1-|c_{i+1/2}|)(\Psi_i-\Psi_{i+1})\right) & \text{else,} \end{cases} \tag{5}$$

$$\Phi^+_{i+1/2} = \alpha_{i+1/2} + \beta_{i+1/2} r^+_{i+1/2}, \quad \Phi^-_{i+1/2} = \alpha_{i+1/2} + \beta_{i+1/2} r^-_{i+1/2}, \tag{6}$$

$$\alpha_{i+1/2} = \frac{1}{2} + \frac{1}{6}(1-2|c_{i+1/2}|), \quad \beta_{i+1/2} = \frac{1}{2} - \frac{1}{6}(1-2|c_{i+1/2}|), \tag{7}$$

$$r^+_{i+1/2} = \frac{\Psi_i - \Psi_{i-1}}{\Psi_{i+1}-\Psi_i}, \quad r^+_{i+1/2} = \frac{\Psi_{i+2}-\Psi_{i+1}}{\Psi_{i+1}-\Psi_i}. \tag{8}$$

TVD advection schemes: Basics III

P_2-PDM, ULTIMATE QUICKEST (Leonard [1979]):

$$\Phi^+_{i+1/2} \to \max\left[0, \min\left(\Phi^+_{i+1/2}, \frac{2}{1-|c_{i+1/2}|}, \frac{2r^+_{i+1/2}}{|c_{i+1/2}|}\right)\right], \quad (9)$$

MUSCL (van Leer [1979]):

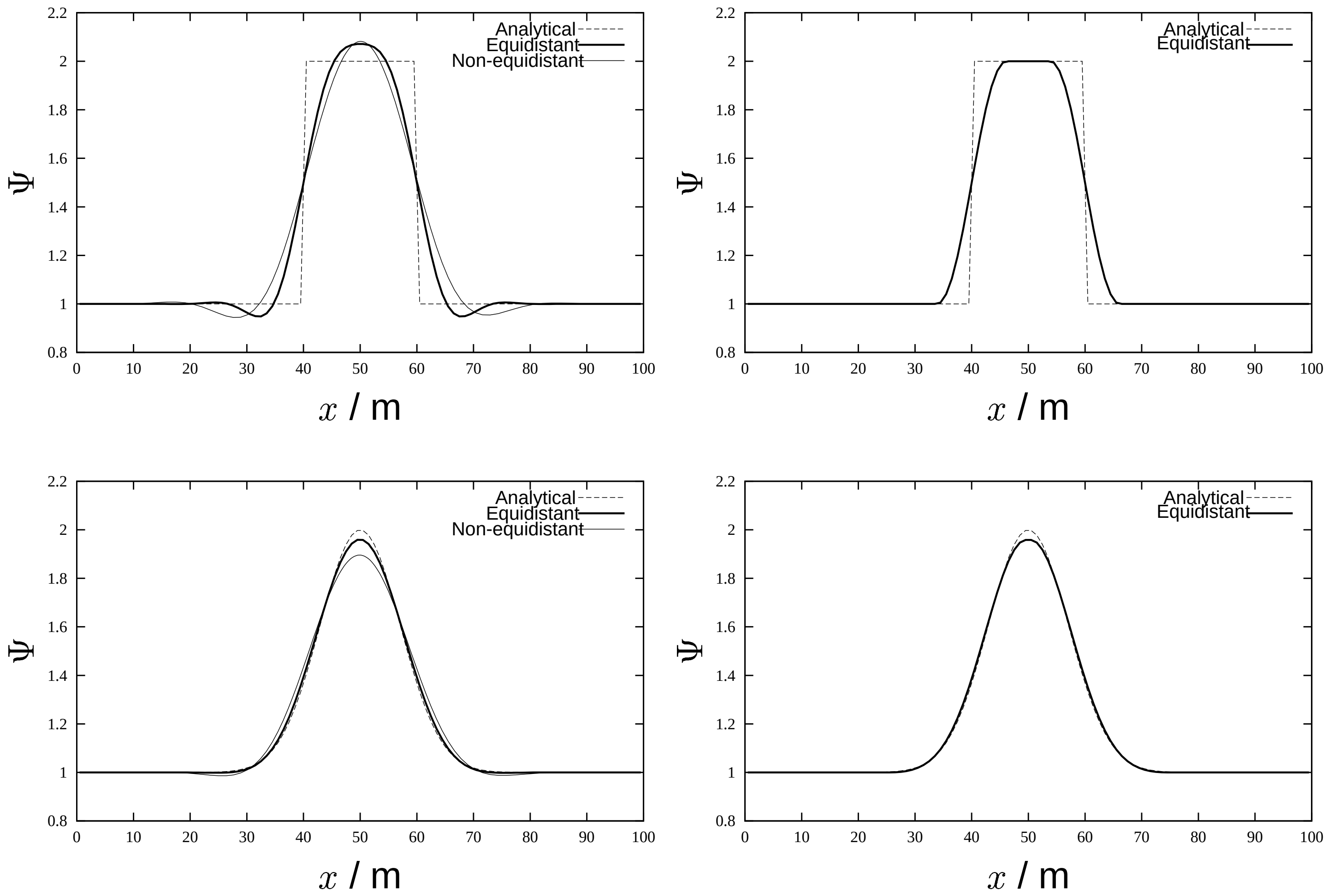
$$\Phi^+_{i+1/2} \to \max\left[0, \min\left(2, 2r^+_{i+1/2}, \frac{1+r^+_{i+1/2}}{2}\right)\right], \quad (10)$$

Superbee (Roe [1985]):

$$\Phi^+_{i+1/2} \to \max\left[0, \min(1, 2r^+_{i+1/2}), \min(r^+_{i+1/2}, 2)\right]. \quad (11)$$

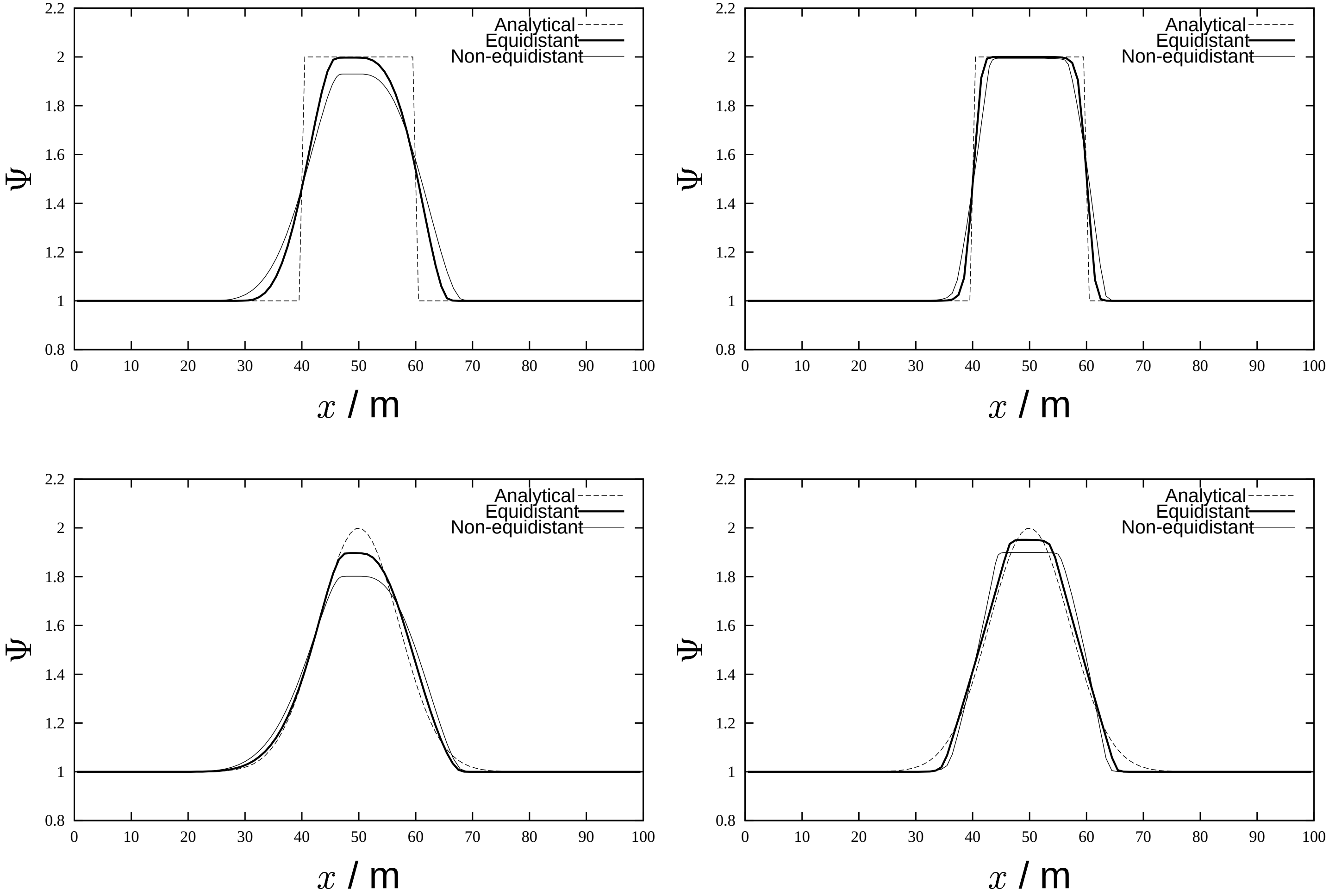
1D test case: P_2 scheme

Left: unlimited; right: limited



1D test: MUSCL & Superbee schemes

Left: MUSCL; right: Superbee



Directional split schemes

$$\partial_t(D\Psi) + \partial_x(uD\Psi) + \partial_y(vD\Psi) = 0 \tag{12}$$

$$\partial_t D + \partial_x(uD) + \partial_y(vD) = 0 \tag{13}$$

Directional split (Pietrzak [1998])

$$D_{i,j}^{n+1/2}\Psi_{i,j}^{n+1/2} = D_{i,j}^{n}\Psi_{i,j}^{n} - \frac{\Delta t}{\Delta x}(D_{i+1/2,j}^{n}F_{i+1/2,j}^{n} - D_{i-1/2,j}^{n}F_{i-1/2,j}^{n}) \tag{14}$$

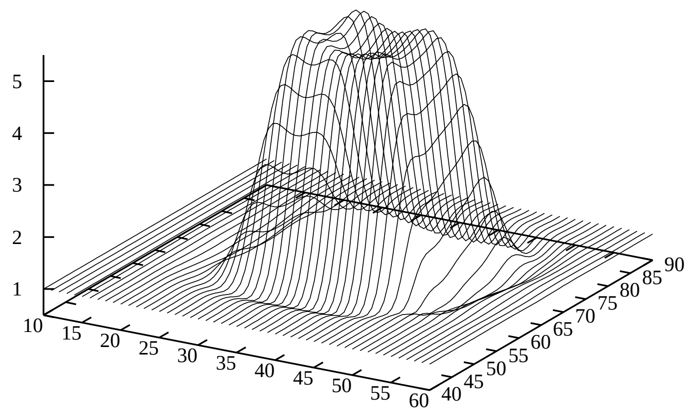
$$D_{i,j}^{n+1}\Psi_{i,j}^{n+1} = D_{i,j}^{n+1/2}\Psi_{i,j}^{n+1/2} - \frac{\Delta t}{\Delta y}(D_{i,j+1/2}^{n}F_{i,j+1/2}^{n} - D_{i,j-1/2}^{n}F_{i,j-1/2}^{n}) \tag{15}$$

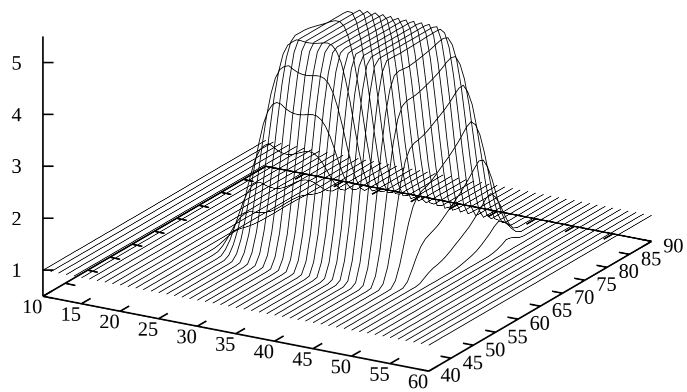
$$D_{i,j}^{n+1/2} = D_{i,j}^{n} - \frac{\Delta t}{\Delta x}(U_{i+1/2,j}^{n} - U_{i-1/2,j}^{n}) \tag{16}$$

$$D_{i,j}^{n+1} = D_{i,j}^{n+1/2} - \frac{\Delta t}{\Delta y}(V_{i,j+1/2}^{n} - V_{i,j-1/2}^{n}). \tag{17}$$

2D test case: P_2 split scheme

Cube resulting after one solid-body rotation with $\Delta x = \Delta y = 1$ m and a Courant number of $c = 0.5$. Left: unlimited P_2 scheme; right: limited P_2-PDM scheme



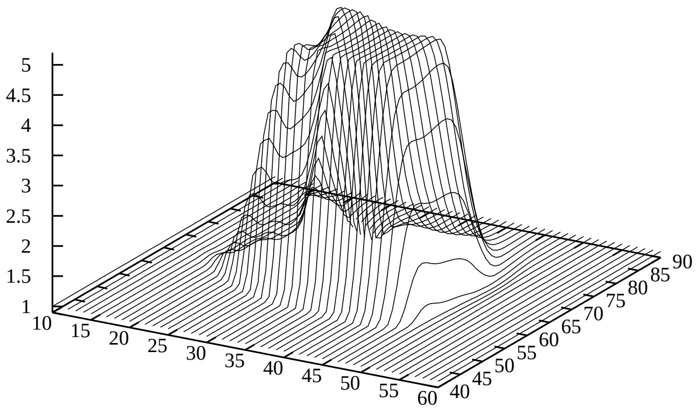


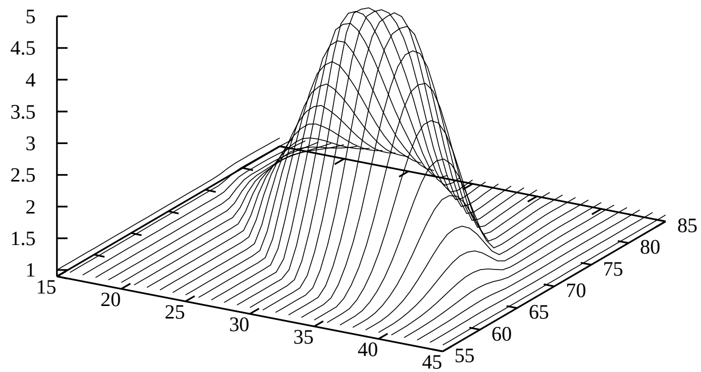
2D test case: Error statistics

Scheme	N	Min	Max	TV/TV_0	TV_{max}/TV_0	L_2 error	$\mathcal{O}$
P_2	100	0.395	5.871	1.2771	1.6331	0.240092	–
P_2	200	0.389	5.868	1.2665	1.6785	0.186427	0.365
P_2	400	0.386	5.867	1.2578	1.7074	0.144827	0.364
MUSCL	100	1.000	5.000	1.0315	1.3003	0.246704	–
MUSCL	200	1.000	5.000	1.0251	1.3456	0.190325	0.374
MUSCL	400	1.000	5.000	1.0172	1.3731	0.147447	0.368
Superbee	100	1.000	5.000	1.0538	1.3093	0.182077	–
Superbee	200	1.000	5.000	1.0395	1.3528	0.129787	0.488
Superbee	400	1.000	5.000	1.0276	1.3784	0.093277	0.477
P_2-PDM	100	1.000	5.000	1.0311	1.3117	0.236535	–
P_2-PDM	200	1.000	5.000	1.0224	1.3536	0.182141	0.377
P_2-PDM	400	1.000	5.000	1.0148	1.3783	0.140810	0.371

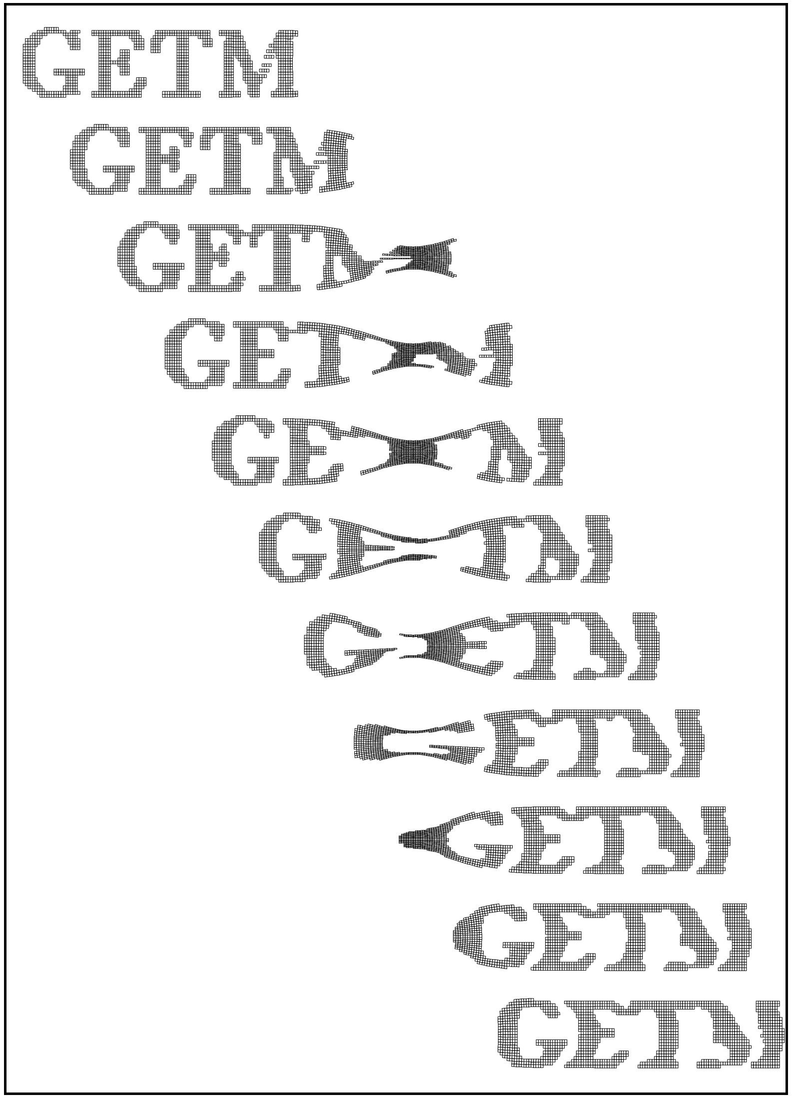
2D test case: Pseudo-2D P_2-PDM

Pseudo-two-dimensional P_2-PDM scheme: Cube (left) and Gaussian bell (right) resulting after one solid-body rotation with $\Delta x = \Delta y = 1$ m and a Courant number of $c = 0.5$



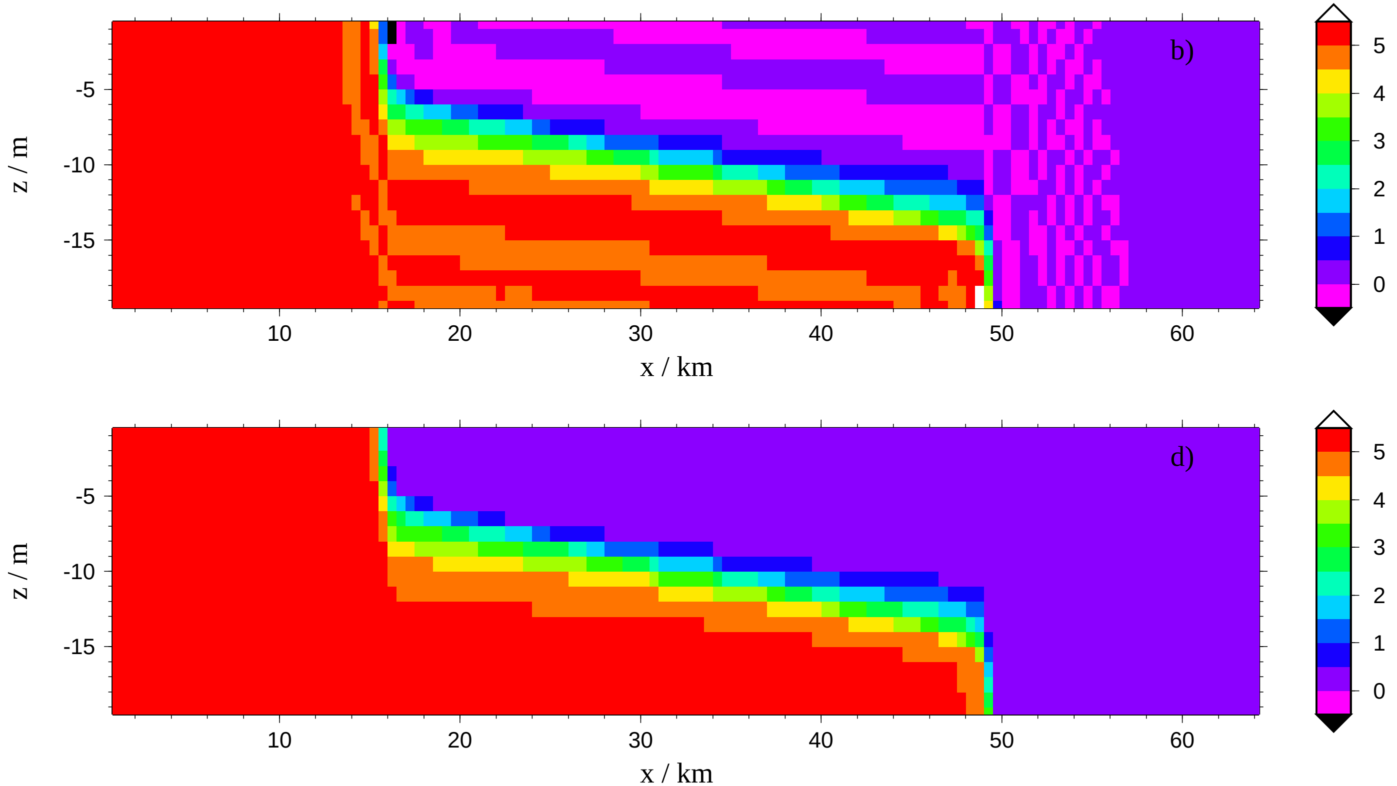


General Estuarine Transport Model



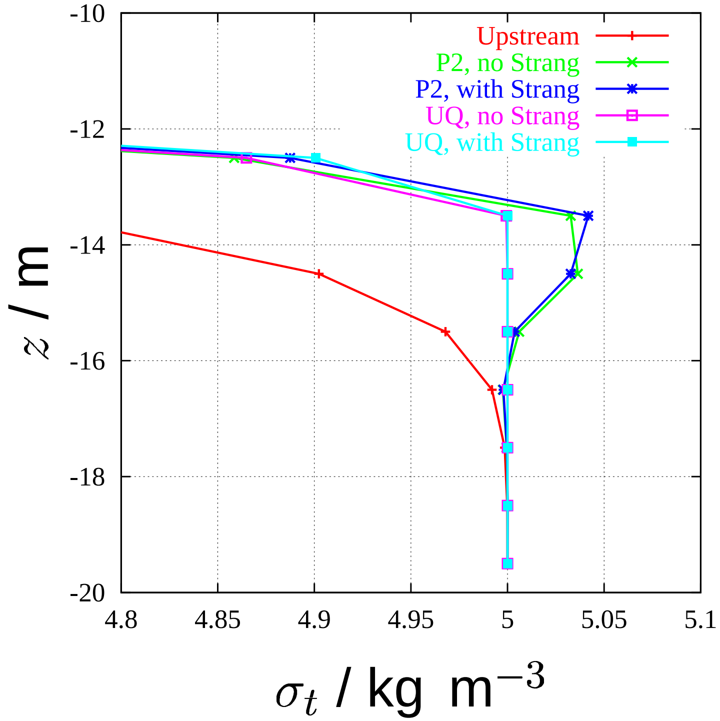
Lock exchange test case I

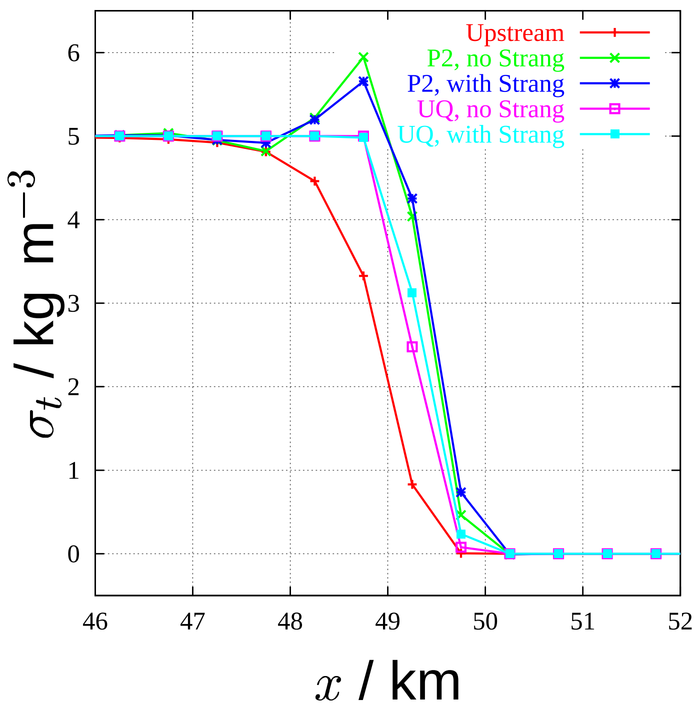
Lock exchange experiment: Density distribution for four different numerical schemes at the end of the experiment. b) P_2 scheme with Strang [1968] splitting; d) limited P_2 scheme with Strang [1968] splitting.



Lock exchange test case II

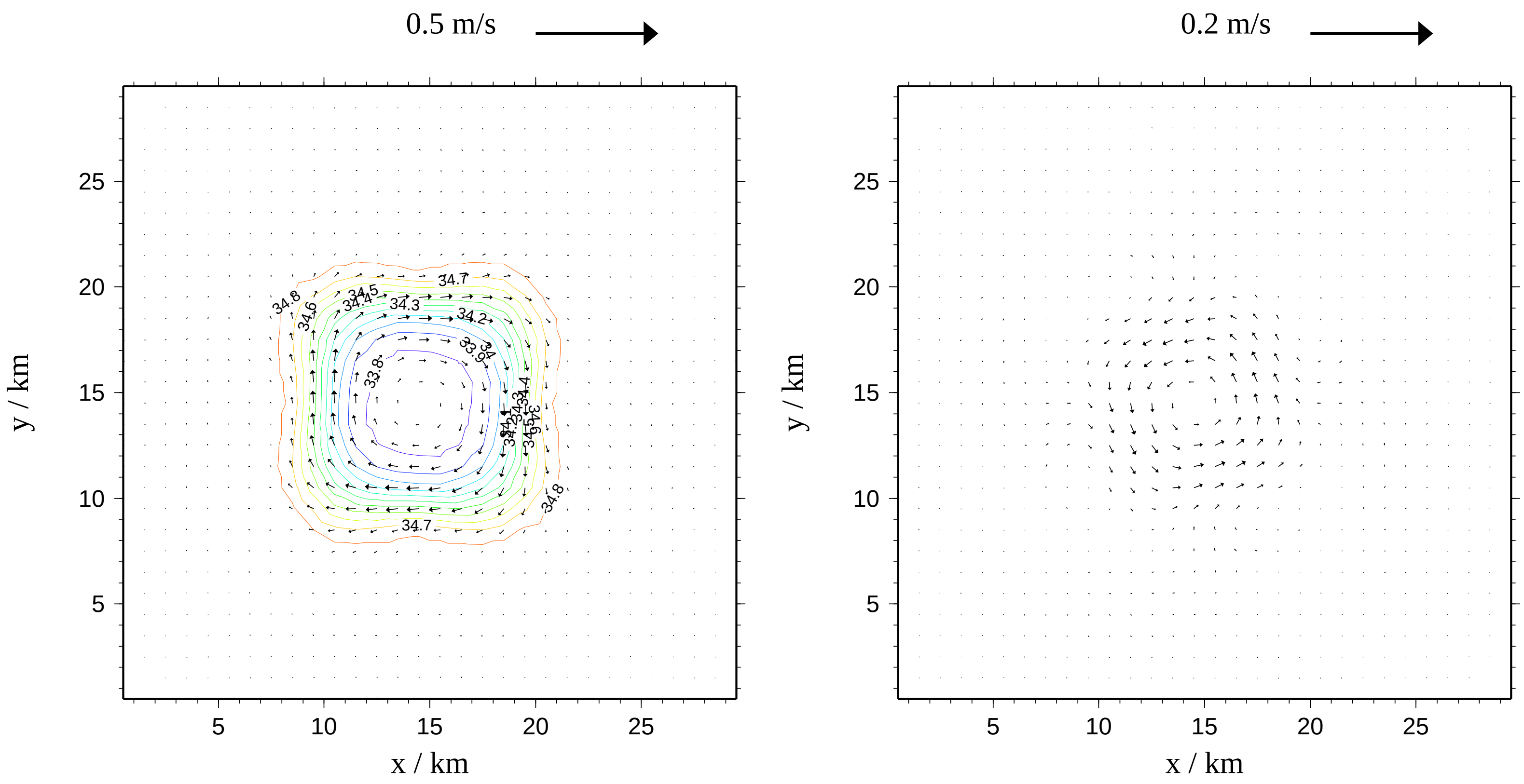
Lock-exchange experiment: vertical density profiles (left) and near-bed density distribution of density (right) for five different numerical schemes (Upstream, P_2, P_2-PDM) at x = 32 km at the end of the experiment:





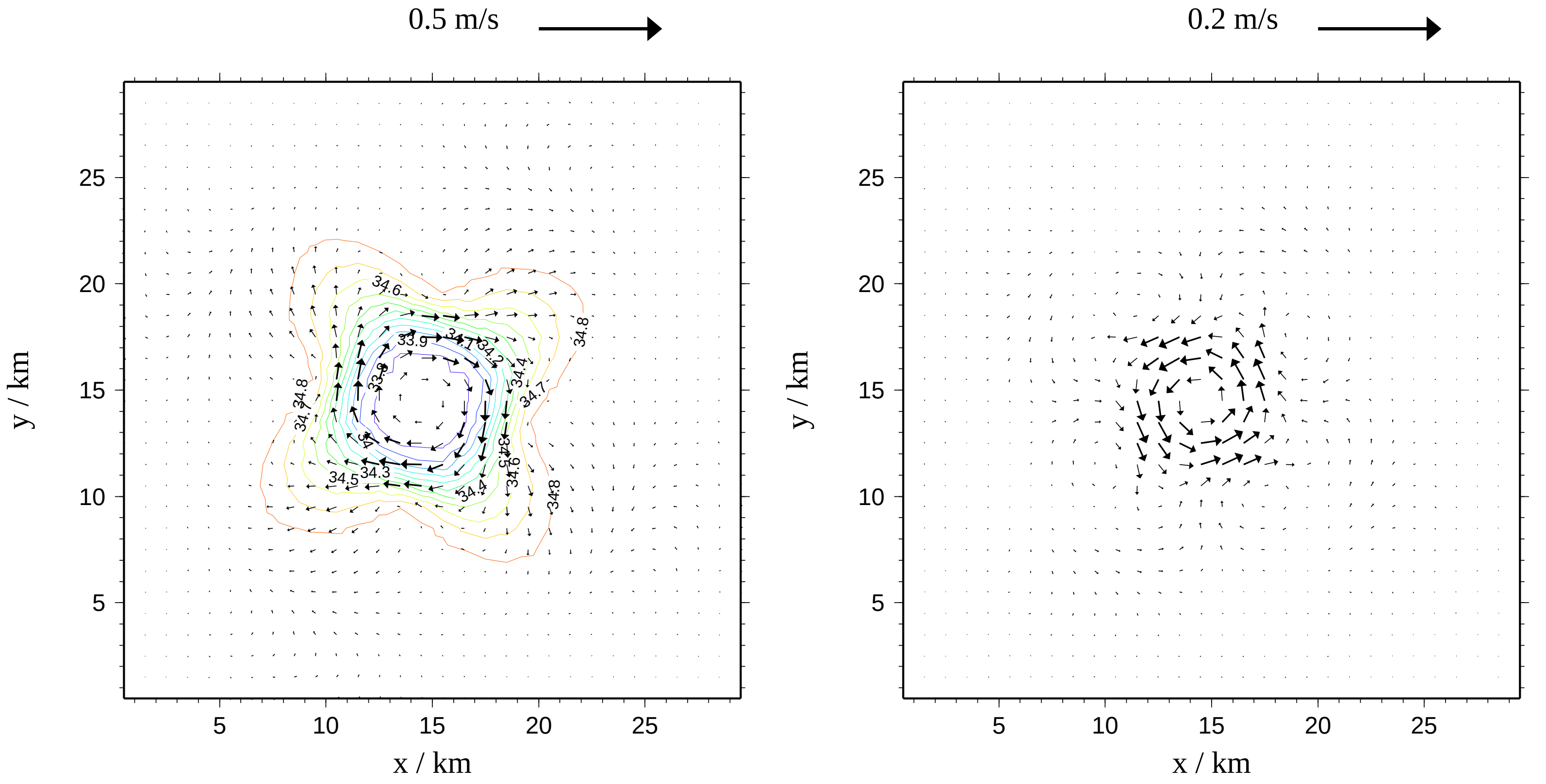
Freshwater eddy I

Left: surface salinity and current vectors; right: bottom current vectors. Momentum advection: multidimensional upwind scheme; salinity advection: TVD-Superbee directional-split scheme.



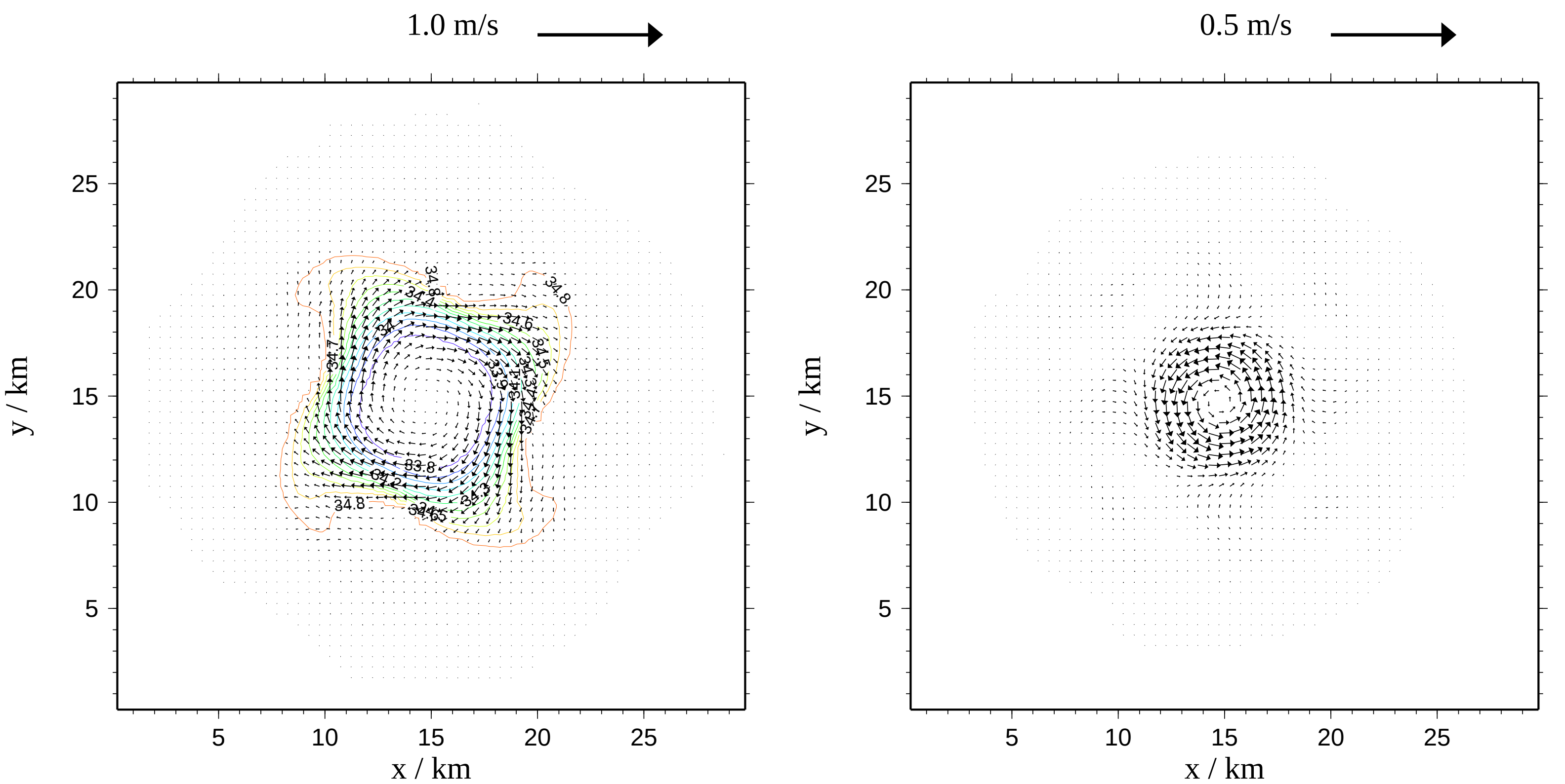
Freshwater eddy II

Left: surface salinity and current vectors; right: bottom current vectors. Momentum advection: momentum and salinity advection: TVD-Superbee directional-split scheme.

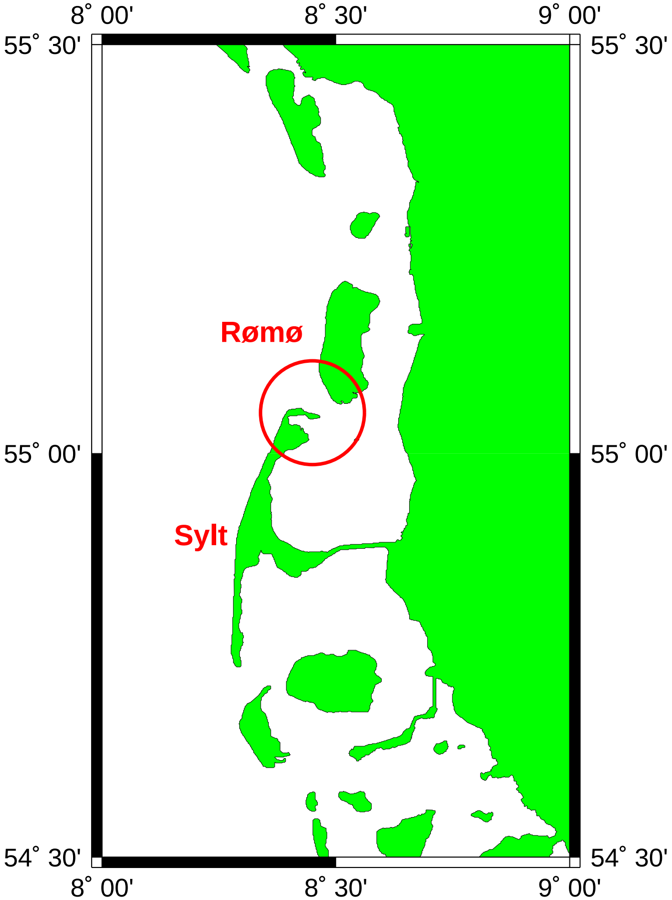


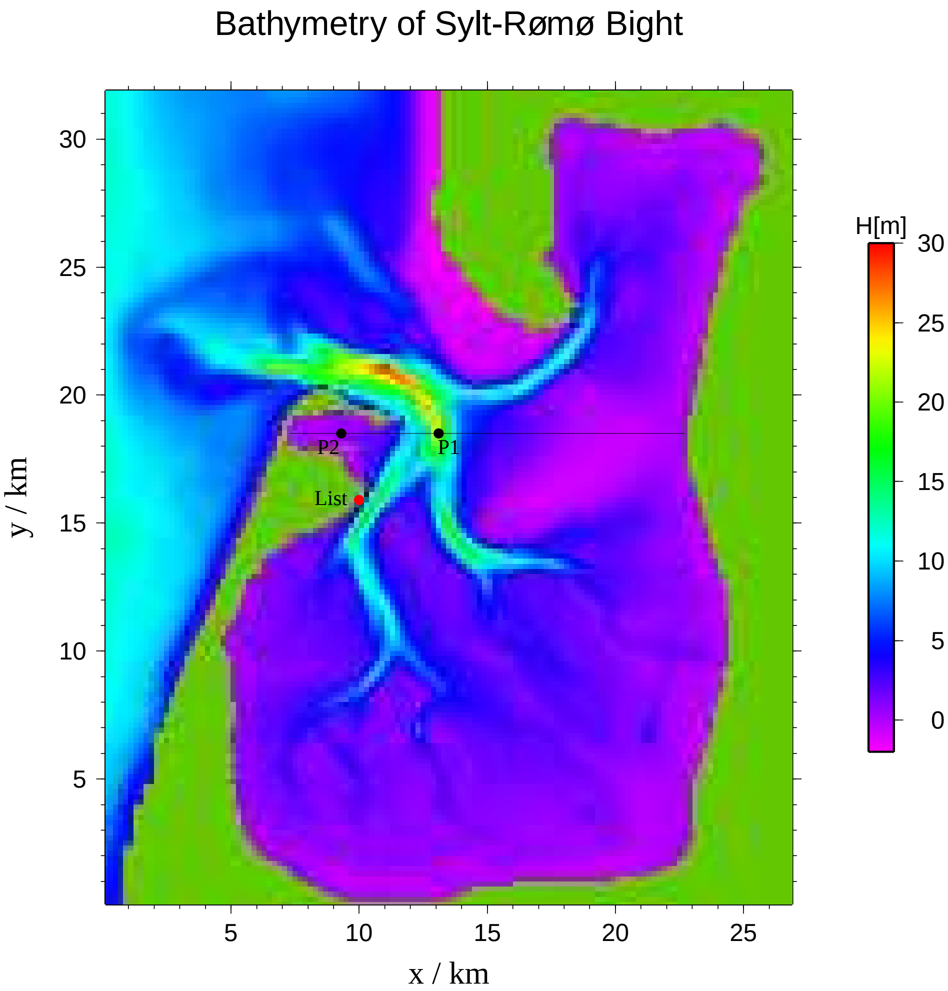
Freshwater eddy III

Left: surface salinity and current vectors; right: bottom current vectors. Momentum advection: momentum and salinity advection: TVD-Superbee directional-split scheme. Increased horizontal resolution with $\Delta x = \Delta y$ = 500 m.



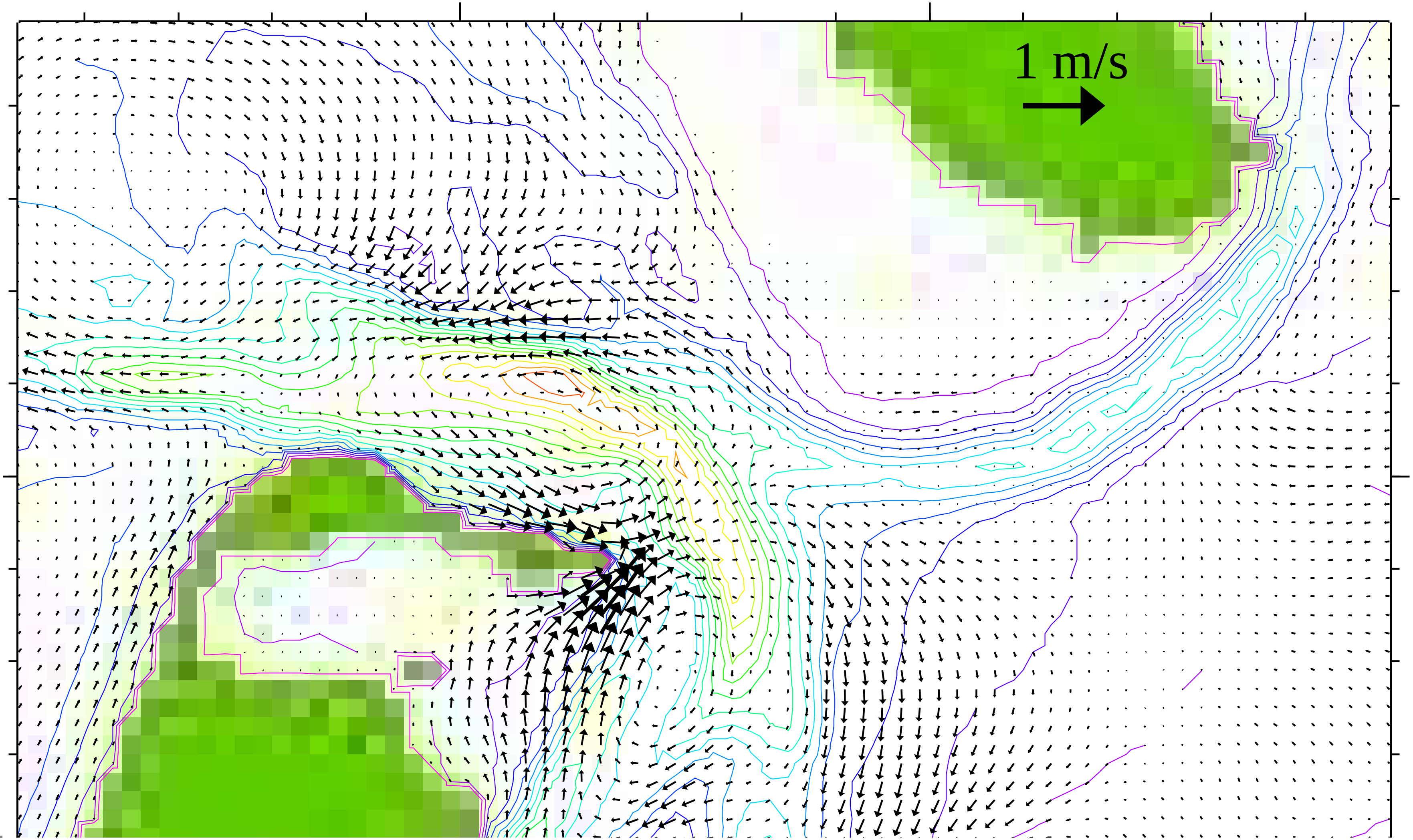
Sylt-Rømø-Bight I





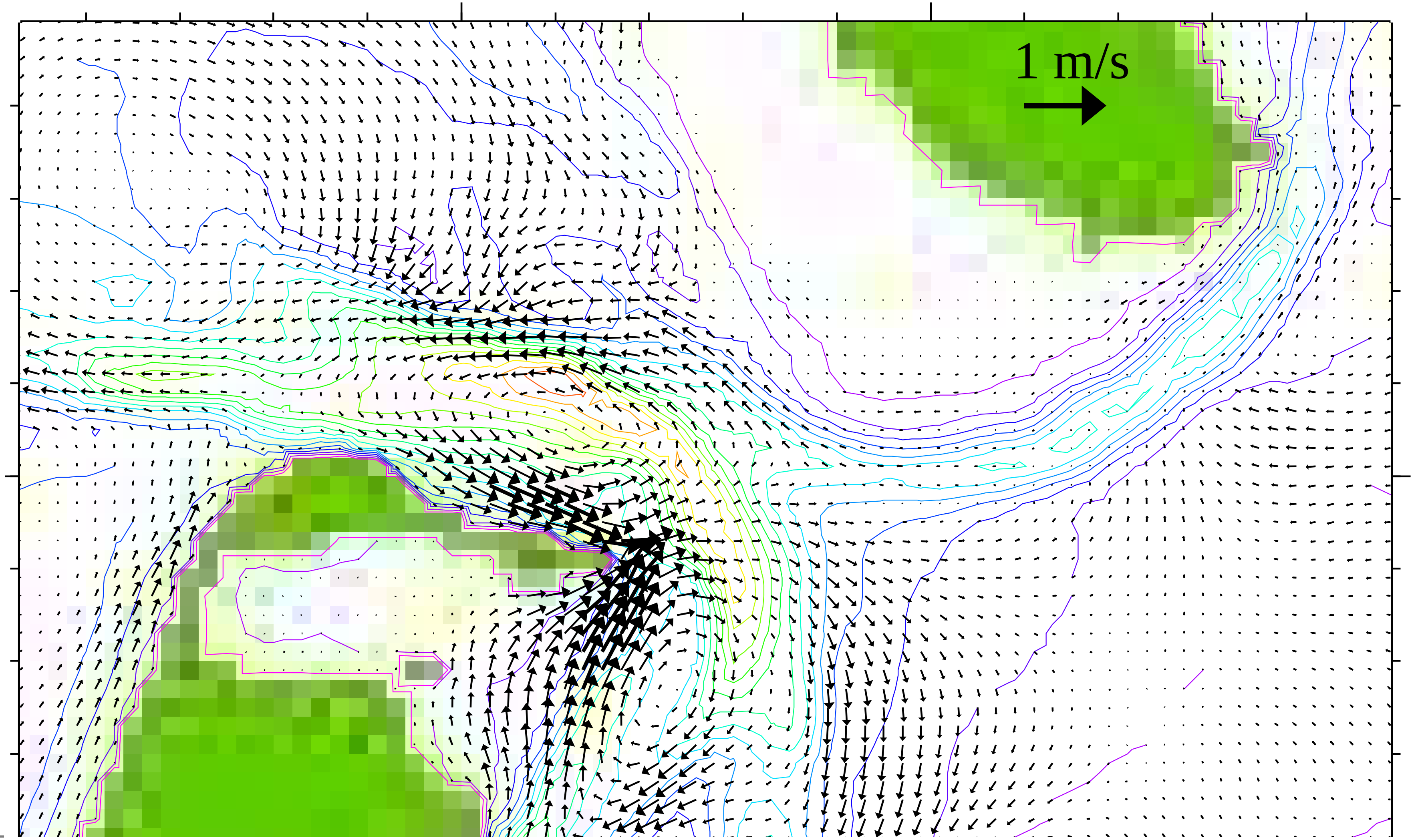
Sylt-Rømø-Bight II

Residual currents, upstream for advection

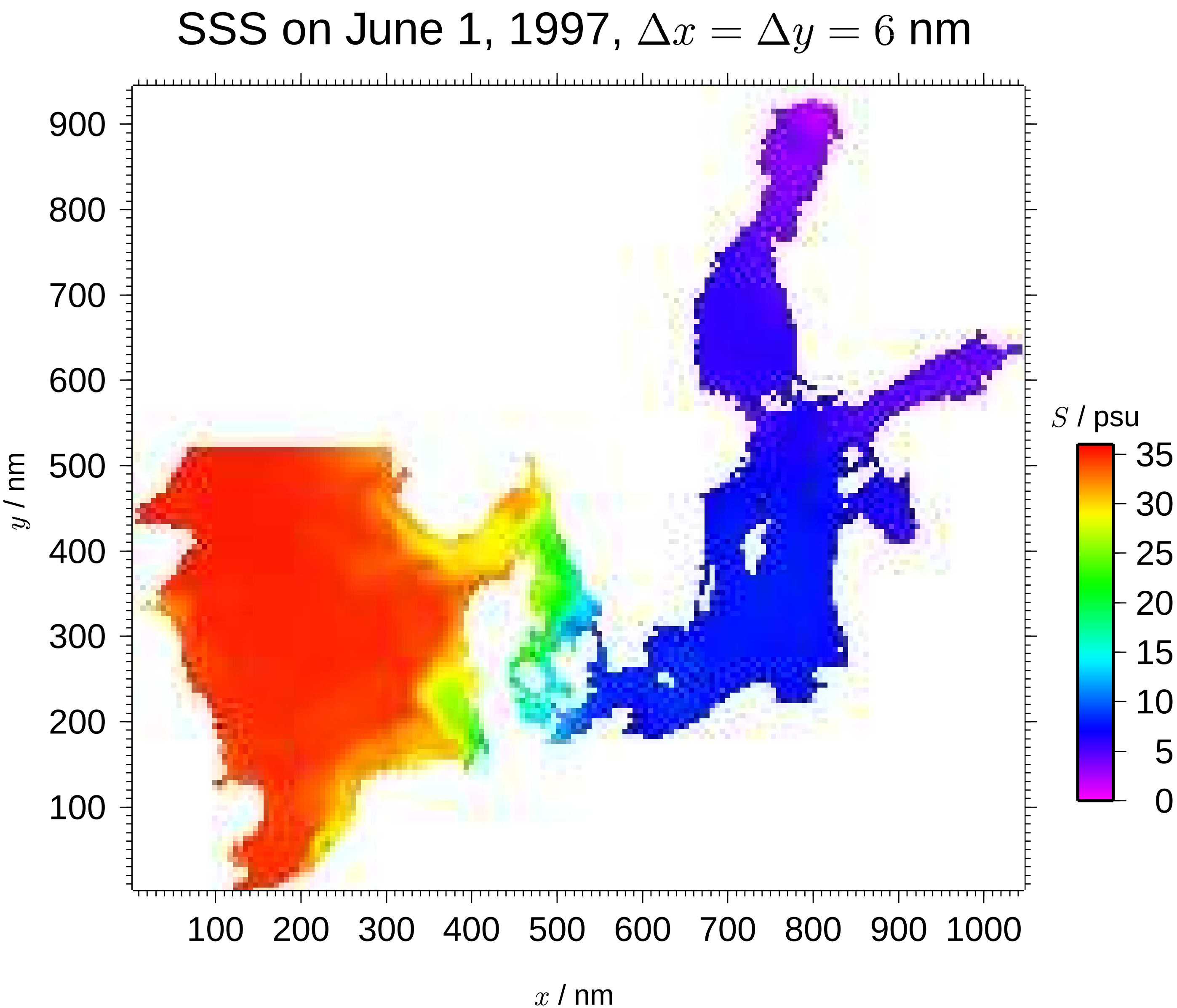


Sylt-Rømø-Bight III

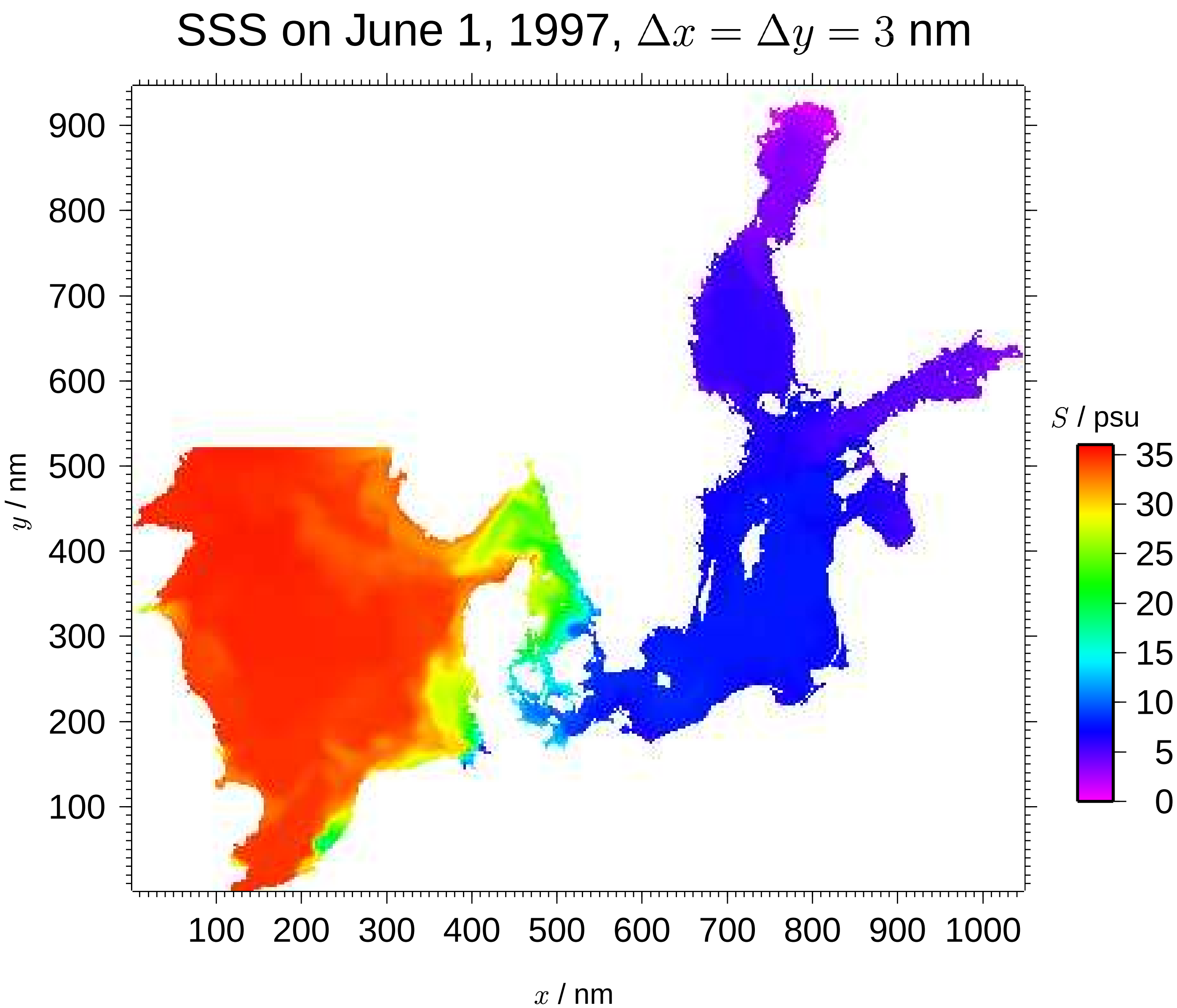
Residual currents, TVD-scheme for advection



North Sea / Baltic Sea I

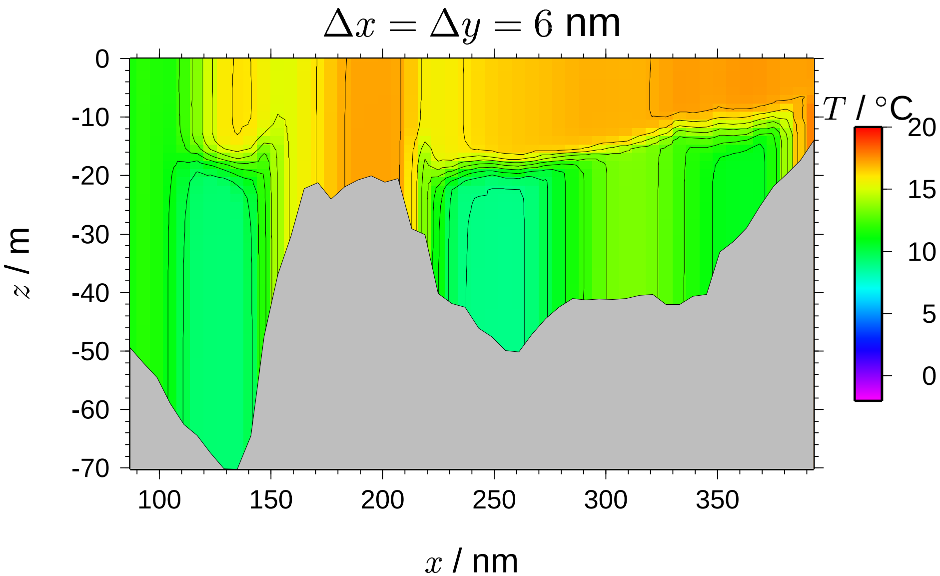


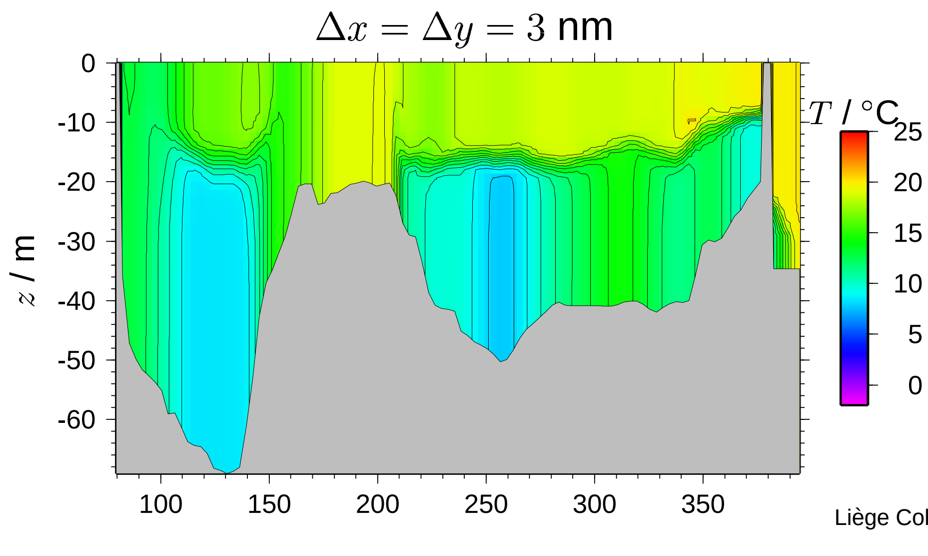
North Sea / Baltic Sea II



North Sea / Baltic Sea III

T across Doggerbank W/E section I, 97/09/01





Conclusions

- Directional split methods allow application of 1D advection schemes in 3D models, but the continuity must be carefully considered
- The monotonicity properties of the 1D TVD schemes are conserved in 3D but not the TVD property (Theorem by Goodman and LeVeque, 1985)
- High-order advection of momentum and turbulent quantities which are staggered on a C-grid, can easily be implemented
- Directional-split TVD advection schemes allow monotone, stable and accurate calculation of advective transport of tracers without the need of stabilising horizontal diffusion