Theoretical Oceanography
Lecture Notes Master AO-W27

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Contents

0

Chapter 1

Literature recommendations

This script is not a textbook, but shall serve as a guide into the field of theoretical oceanography. The discipline in the foreground is physics. We demonstrate how the complex phenomena of oceanic flow can be derived and understood from basic laws of hydrodynamics and thermodynamics. Hence, textbooks on "descriptive oceanography" are of limited help for this purpose, but may serve as an excellend supplement, since a detailed description of observations cannot be given in the limited frame of this lecture. Here the book *Descriptive Physical Oceanography* of L. Talley et al. [7] is strongly recommended. This book provides impressive insight in the beauty of our earth not to be gained with pure eyes. Also the book of M. Tomczak and J. Godfrey [8] gives a good overview. There is also a web based version, http://www.es.flinders.edu.au/~mattom/regoc/pdfversion.html.2

The lecture does not follow a specific textbook. Many sections are inpired by the recently published book *Ocean Dynamics* of D. Olbers, J. Willebrand and C. Eden, [6]. The analytical theory based on formal solutions of linear partial differential equations (Green functions) follows W. Fennel and H. U. Lass, [2]. Stimulating ideas but also some of the figures are from previous lectures of Wolfgang Fennel.

The books of

- G. Vallis, *Atmosphere and Ocean Fluid Dynamics*, [9],
- J. Apel, *Principles of Ocean Physics*, [1],
- A. Gill, *Atmosphere-Ocean Dynamics*, [3],
- P. Kundu, *Fluid Mechanics*, [4]
- J. Lighthill, *Waves in Fluids*, [5]

are also to be recommended. This is only a minor selection.

For a deeper understanding of the principles of small scale hydrodynamics, see any of the numerous textbooks on theoretical physics.

Chapter 2

Basic equations

The fluid water in the ocean or the gaseous air are continuous deformable bodies. Here we ask for macroscopic properties and mostly we do not consider the microscopic fundamentals. So water and air have given thermodynamic properties like density, salinity, compressibility, specific heat or viscosity. This suggests that the field of thermodynamics should be a basic ingredient of theoretical oceanography. Water in the ocean is moving and we will base our consideration on the appropriate theory, the field of hydrodynamics. Both branches of theoretical physics will not be introduced here, but only those essential needed to understand ocean or atmosphere physics will be repeated shortly on an introducing level.

2.1. Flow kinematics

This section introduces the basic kinematic variables suitable to describe the motion of a fluid. We use mostly geophysical co-ordinates on a sphere or a local Cartesian co-ordinate system. Eulerian and Lagrangian representation are introduced. The Lagrangian view is more appropriate when considering the spreading of water masses within the ocean, the Eulerian view is mathematically simpler for most applications, especially for numerical ocean or atmosphere models.

2.1.1. Co-ordinate systems

For two persons who want to meet somewhere on the earth it should be sufficient to exchange the co-ordinates of the meeting point and to find a way to navigate to this place. As long as they meet at the solid or fluid surface of the earth, the two geographical co-ordinates longitude λ and latitude φ should be sufficient to arrange a meeting. If diving into the ocean or flying into the air is allowed, a vertical co-ordinate z is needed too. Not to forget that both should be on the spot at the same time t. Assuming that longitude λ is positive eastward, latitude φ is positive northward and height z orients positive upward, the three spatial co-ordinates form a right handed system. In oceanography it is common to count z negative below the sea surface.

So simple this concept seems to be so complex it is. The earth is not a sphere but is deformed from inhomgeneous mass distribution, earth rotation and tides. The ocean surface is mostly elevated from its equilibrium position by winds and non-uniform heating. At longer time scales the average sea level is changing, continents are drifting and the geophysical co-ordinate system is well defined only by a complex procedure. Such a general co-ordinate system is not suitable for an introduction into basic theoretical concepts. Hence, for global considerations we approximate the earth by a sphere which is sufficient for this lecture. In most cases even a local Cartesian co-odinate system $\mathbf{x} = (x, y, z)$ is used.

It should be mentioned that other systems than geographical co-ordinates may be useful for numerical models of the ocean or the atmosphere. Especially the depth z as vertical co-ordinate may be replaced by another co-ordinate, for example the relative depth in relation to the total depth or by the water density. Advantages and disadvantages of a special choice is beyond the scope of the lecture.

2.1.2. Eulerian and Lagrangian representation

Euler [1] and Lagrange [2] developed a complementary approach to describe the motion of fluids.

Lagrange considers fluid elements, marked by a vector of properties, **a**. It is located at position **x**, fluid motion is the change of its position with time,

$$\mathbf{x} = \mathbf{x}(\mathbf{a}, t). \tag{2.1}$$

A possible way to mark a fluid element is its position at time t_0,

$$\mathbf{x}(\mathbf{a}, t_0) = \mathbf{a}. \tag{2.2}$$

The Lagrangian velocity a is

$$\mathbf{u}^L(\mathbf{a}, t) = \frac{\partial}{\partial t}\mathbf{x}(\mathbf{a}, t). \tag{2.3}$$

In turn the position of the fluid elements at time, t, can be found if the velocity of all fluid elements is known,

$$\mathbf{x}(\mathbf{a}, t) = \mathbf{x}(\mathbf{a}, t_0) + \int_{t_0}^{t} dt' \mathbf{u}^L(\mathbf{a}, t'). \tag{2.4}$$

$\mathbf{u}^L$ is only a function of time! Hence, the varying set of positions of fluid elements in principle describes the fluid motion.

[1] Leonhard Euler (* 15. April 1707 in Basel; †7. September 1783 in Sankt Petersburg. Swiss mathematician and physicist. Basel, Berlin, Sankt Petersburg.

[2] Joseph-Louis de Lagrange (* 25. January 1736, Turin (Giuseppe Lodovico Lagrangia); †10. April 1813, Paris. Italian Mathematician. Turin, Berlin, Paris.

This representation has little use for practical applications. Marking a fluid element is difficult (aspects of quantum statistics that elementary particles are indistinguishable need consideration). Much simpler is the idea to consider the velocity of the fluid as function of position, $\mathbf{x}$, and time t,

$$\mathbf{u} = \mathbf{u}^E(\mathbf{x}, t). \tag{2.5}$$

This velocity field is called Eulerian velocity. With modern ADCPs or LASER-Doppler anemometers snapshots of the velocity field can be gained.

Both representations of motion are equivalent. At time t a fluid element marked with $\mathbf{a}$ with $\mathbf{x} = \mathbf{x}(\mathbf{a}, t)$ is found at $\mathbf{x}$. The local motion is the velocity of the fluid elements, hence,

$$\mathbf{u}^E(\mathbf{x}, t) = \mathbf{u}^L(\mathbf{a}, t). \tag{2.6}$$

The task to find the trajectory of a fluid element, $\mathbf{X}(t)$, requires the solution of the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{X} &= \mathbf{u}^E(X, t), \\ \mathbf{X}(t_0) &= \mathbf{a}. \end{aligned} \tag{2.7}$$

The equivalence of Eulerian and Lagrangian view applies for all quantities, say A,

$$A^E(\mathbf{X}(\mathbf{a}, t), t) = A^L(\mathbf{a}, t). \tag{2.8}$$

The time tendency of A within the moving fluid element reads

$$\begin{aligned} \frac{\partial A^L(\mathbf{a}, t)}{\partial t} &= \left.\frac{\partial A^E(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}} + \left.\frac{\partial A^E(\mathbf{X}, t)}{\partial \mathbf{X}}\right|_t \cdot \frac{\partial \mathbf{X}}{\partial t} \\ &= \left.\frac{\partial A^E(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}} + \mathbf{u}^E(\mathbf{x}, t) \cdot \boldsymbol{\nabla}\, A^E(\mathbf{X}, t)\big|_t . \end{aligned} \tag{2.9}$$

Defining the derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}^E \cdot \boldsymbol{\nabla}, \tag{2.10}$$

we get

$$\frac{D}{Dt}A^E(\mathbf{X}, t) = \frac{\partial}{\partial t}A^L(\mathbf{a}, t). \tag{2.11}$$

This derivative at the right hand side describes the time tendency of a property A on the path of a fluid element, $\mathbf{a}$. It is called material or substantial derivative. In the Eulerian view, left hand side, the material derivative has two contributions, the time tendency at fixed position $\mathbf{x}$ and the advective (engineering calls it convective) time tendency from the advection of fluid elements with different properties to the position $\mathbf{x}$.

For the special case

$$\frac{\partial}{\partial t} A^L(\mathbf{a}, t) = 0, \tag{2.12}$$

or equivalently in the Eulerian picture

$$\frac{D}{Dt} A^E(\mathbf{X}, t) = 0, \tag{2.13}$$

the property A is constant at the trajectory. In this case A is called a *conservative quantity*.

Exercise 2.1 Consider a river with a cold spring and assume constant solar warming of the water with 300 Wm^{-2}. The flow velocity should be 1 ms^{-1}. To keep the example simple, the depth of the river should be 1 m, the water column is mixed permanently so that the water temperature stays vertically uniform. Water temperature is essential for observers, they need to learn the laws of temperature change. Find an appropriate description of the water temperature for three observers:

- an observer traveling along the river shore and measuring the temperature occasionally,
- a shortsighted observer living on a float drifting downstream with the flow measuring the temperature occasionally,
- a shortsighted observer navigating with a boat up and downstream using the thermometer frequently. The boat has a log to measure the speed through the water too.

"Shortsighted" means, that the observer cannot see neither the riverside nor the river floor. Think about appropriate coordinates and descriptions. Relate them to the Lagrangian or Eulerian view!

Both views are equivalent, but the Lagrangian approach becomes inconsistent in the turbulent case. Here the definition of a fluid element becomes difficult and trajectories starting from almost identical positions may diverge exponentially since the Ljapunov-exponent is positive. For these and other reasons we use mostly the Eulerian approach.

Exercise 2.2 Find examples for technical applications or scientific questions in oceanography where either the Lagrangian or the Eulerian view is preferable.

Stream lines

The field of tangent curves of a velocity field $\mathbf{u}(\mathbf{x}, t)$ at time t are called *streamlines.* They may be defined as vector product

$$\mathbf{u} \times d\mathbf{x} = 0. \tag{2.14}$$

An equivalent definition reads

$$\frac{dx}{u(\mathbf{x},t)} = \frac{dy}{v(\mathbf{x},t)} = \frac{dz}{w(\mathbf{x},t)}. \tag{2.15}$$

For a steady flow, i.e., when the Eulerian velocity is independent of time, streamlines coincide with the trajectories of fluid elements.

Rotation and deformation of fluid elements

At this point the kinematic of rotation and deformation of fluids should be repeated in detail. This is not done here, but it is strongly suggested to repeat the corresponding section in textbooks on hydrodynamics. Here we consider only some examples, needed later in the discussion of friction.

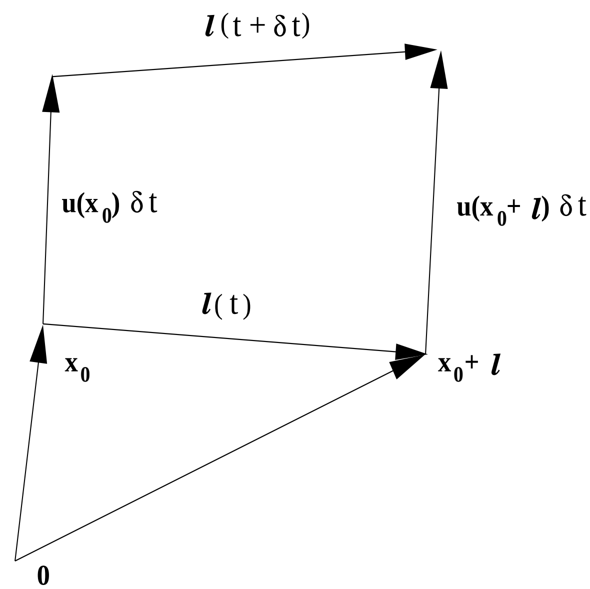


Figure 2.1. Deformation and rotation of a length vector $\mathbf{l}$. The fluid element at $\mathbf{x}_0$ moves with a different velocity than that at $\mathbf{x}_0 + \mathbf{l}$. After time δt the vector appear stretched and rotated.

We use a Lagrangian view and consider two Lagrangian fluid elements separated at time t_0 by a distance vector $\mathbf{l}$. Hence, the positions are $\mathbf{x}_0$ and $\mathbf{x}_0 + \mathbf{l}$. The fluid elements are moving with velocity $\mathbf{u}(\mathbf{x}_0, t)$ and $\mathbf{u}(\mathbf{x}_0 + \mathbf{l}, t)$. We ask for the time tendency of the distance *vector* $\mathbf{l}$, which follows from the relative velocity $\delta\mathbf{u}$ of both fluid elements,

$$\begin{aligned}\mathbf{l}(t+\delta t) &= \mathbf{l}(t)+\delta\mathbf{u}\delta t\\ &= \mathbf{l}(t)+\left(\mathbf{u}(\mathbf{x}_0+\mathbf{l},t)-\mathbf{u}(\mathbf{x}_0,t)\right)\delta t\\ &\approx \mathbf{l}(t)+\mathbf{l}(t)\cdot\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0,t)\delta t+\mathcal{O}(l^2).\end{aligned} \tag{2.16}$$

Hence, we have derived again the Lagrangian time derivative,

$$\frac{D\mathbf{l}}{Dt} = \mathbf{l}(t)\cdot\boldsymbol{\nabla}\mathbf{u}(\mathbf{x}_0,t). \tag{2.17}$$

So far nothing is new.

Now we split the velocity gradient into a symmetric and an anti-symmetric contribution,

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right), \tag{2.18}$$

and introduce the abbreviations

$$\begin{aligned}D_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right) && \text{deformation tensor,}\\ R_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right) && \text{rotation tensor.}\end{aligned} \tag{2.19}$$

The names suggest that fluid motion consists of deformation and rotation represented by two tensors build by the velocity derivatives,

$$\frac{D\mathbf{l}}{Dt} = (D+R)\cdot\mathbf{l}(t). \tag{2.20}$$

The rotation tensor has zero diagonal elements and is anti-symmetric, Hence it is defined by three independent components ω_i only,

$$(R_{ij}) = \frac{1}{2}\begin{pmatrix}0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{pmatrix}. \tag{2.21}$$

The elements ω_i form a pseudo-vector,

$$\begin{aligned}\boldsymbol{\omega} &= (\omega_1,\omega_2,\omega_3)\\ &= \left(\frac{\partial u_3}{\partial x_2}-\frac{\partial u_2}{\partial x_3},\frac{\partial u_1}{\partial x_3}-\frac{\partial u_3}{\partial x_1},\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right)=\boldsymbol{\nabla}\times\mathbf{u}\end{aligned} \tag{2.22}$$

The resulting evolution of a distance vector reads

$$\begin{aligned}\frac{D\mathbf{l}}{Dt} &= (D+R)\cdot\mathbf{l}\\ &= D\cdot\mathbf{l}+\frac{1}{2}\left(\boldsymbol{\omega}\times\mathbf{l}\right)\\ &= D\cdot\mathbf{l}+\frac{1}{2}\left(\boldsymbol{\nabla}\times\mathbf{u}\right)\times\mathbf{l}.\end{aligned} \tag{2.23}$$

It is interesting to note, that the time tendency of several quantities is influenced by deformation or rotation only. To make the above statement more clear, we consider the distance $l = |\mathbf{l}|$ and the volume V as examples. Intuitively it should be clear, that rigid rotation should neither change distances of fluid elements nor volumes. We show this by considering the time tendencies of l and V. For the distance we get

$$\frac{Dl^2}{Dt} = 2\mathbf{l} \cdot (D \cdot \mathbf{l} + R \cdot \mathbf{l}). \tag{2.24}$$

It is easy to show that

$$\mathbf{l} \cdot R \cdot \mathbf{l} = 0. \tag{2.25}$$

Hence, rotation does not change the distance of Lagrangian fluid elements.

The volume of a fluid element (assume a parallelepiped) with edges **a**, **b** and **c** reads

$$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}, \tag{2.26}$$

and the time tendency results from changes of the vectors **a**, **b** and **c**,

$$\begin{aligned}\frac{DV}{Dt} &= \left(\frac{D\mathbf{a}}{Dt} \times \mathbf{b}\right) \cdot \mathbf{c} + \left(\mathbf{a} \times \frac{D\mathbf{b}}{Dt}\right) \cdot \mathbf{c} + (\mathbf{a} \times \mathbf{b}) \cdot \frac{D\mathbf{c}}{Dt}, \\ &= (D \cdot \mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} + (\mathbf{a} \times D \cdot \mathbf{b}) \cdot \mathbf{c} + (\mathbf{a} \times \mathbf{b}) \cdot D \cdot \mathbf{c}\end{aligned} \tag{2.27}$$

the terms containing R cancel out. Additionally, Eq. 2.27 can be rewritten so that only the trace of D contributes to the volume evolution,

$$\frac{DV}{Dt} = V D_{ii} = V \boldsymbol{\nabla} \cdot \mathbf{u}. \tag{2.28}$$

As a first result this means, that a divergent flow goes along with changing fluid element volumes. A positive divergence means expansion, a negative divergence means compression. For later reference we note, that a non-divergent flow corresponds to volume conservation, such a flow is called incompressible. For most application in ocean and atmosphere physics this is an excellent approximation and helps to simplify theories considerably.

Recall that the trace of a tensor is invariant against orthogonal co-ordinate transformation. In any case a coordinate system can be found, where the tensor is represented by the set of eigenvalues, $\lambda^{(i)}$,

$$D = (D_{ij}) = \begin{pmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{pmatrix}. \tag{2.29}$$

Hence, the volume change is related only to the flow divergence which in turn is fully defined by the sum of the eigenvalues of D, $\lambda^{(i)}$. This will be used later when the shape of the stress tensor will be defined.

Exercise 2.3 Verify Eq.s 2.23, 2.25, 2.27 and 2.28!

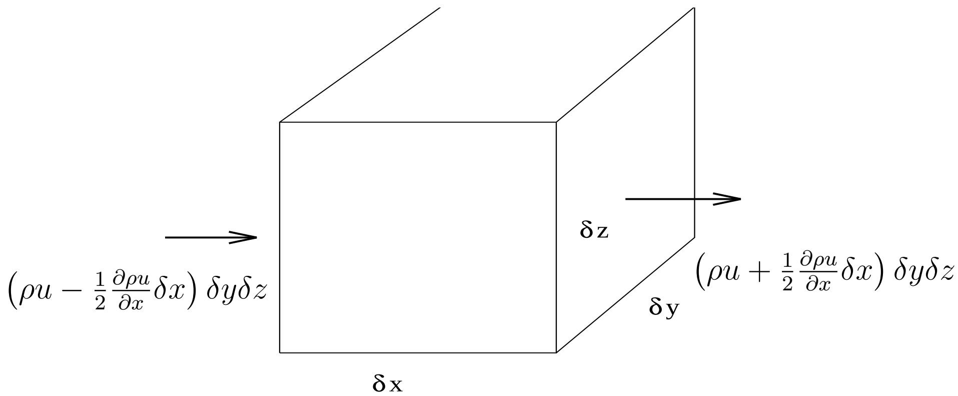


Figure 2.2. Budget of mass for a flow through a volume element

What we should know now?
Fluid motion is described either

- *by the field of time dependent velocity vectors at fixed positions (Eulerian view),*
- *by the field of varying position of marked fluid elements (Lagrangian view).*

2.2. Conservation laws and balance equations

We introduce the concept of fluxes through the surface of a fluid element as genuine source of changes of a quantity. The basic input is the conservation of the number of atoms and of its genuine properties. The mass equivalent of the binding energy from chemical reactions is negligibly small compared with the total mass.

2.2.1. Mass conservation

Consider the flow of a fluid through a volume element of fixed size, position and shape (See Fig. 2.2, Eulerian view). The volume is $\delta V = \delta x \delta y \delta z$, the mass within the volume element is δM and the density $\rho = \frac{\delta M}{\delta V}$. Fig. 2.2 shows fluxes in x-direction, all distances are small and a linear Taylor decomposition of the flux through the $y-z$-surface ρu applies. The fluxes in the other directions can be treated similarly. As long as there is no source or sink of atoms within the volume element, all changes of mass are due to surface fluxes,

$$d\delta M = -\left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right)\delta x \delta y \delta z dt = -\boldsymbol{\nabla}\cdot(\mathbf{u}\rho)\,\delta V dt. \tag{2.30}$$

Since the volume δV is independent of time, the density change is

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\mathbf{u}\rho)\,. \tag{2.31}$$

or equivalently

$$\frac{D\rho}{Dt} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0. \tag{2.32}$$

This is one of the basic conservation laws in hydrodynamics and will be used repeatedly later.

Note, the assumption of a sufficiently small volume element does not limit the generality of this result, since a finite volume element can be arbitrarily subdivided into smaller elements.

An alternative approach starts from the equivalence of the change of mass (left hand side) and mass fluxes through the surface (right hand side)

$$-\frac{\partial}{\partial t} \int_{V_0} \rho dV = \oint_{S_0} \rho \mathbf{u} \cdot d\mathbf{f}. \tag{2.33}$$

$\rho \mathbf{u} \cdot d\mathbf{f}$ is the mass flux through a surface element counted positive for outward flow and negative for inward flow. With the Gaussian theorem the surface integral over the fluxes can be written as a volume integral over the flux divergence,

$$-\frac{\partial}{\partial t} \int_{V_0} \rho dV = \int_{V_0} \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) dV. \tag{2.34}$$

Since the volume is fixed, but arbitrary, this identity must hold for any sub-volume under the integral or again,

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\mathbf{u}\rho)\,. \tag{2.35}$$

2.2.2. Salinity

Oceans do not consist only of water but contain a large amount of dissolved substances - mostly chloride, sodium (responsible for the salty taste), sulfate, magnesium and many other cations and anions. Notably the total amount of salt may vary considerably, the relative amount of ions itself varies only slightly throughout the different ocean areas. This sheds some light on the velocity of salt cycling and the cycling of fresh water - the intensity of sources or sinks of salt ions is much weaker than the intensity of the fresh water cycling - mainly driven by evaporation and precipitation. For our purposes it is sufficient to consider only "salt", the distinction of individual ions is not needed. The concentration of salt s is given as a mass concentration, $s = m_s/m$ which is independent of thermal expansion of the ocean. Usually not the salt concentration, but "salinity" $S = 1000s$ is used for simplicity. The (volume) density, ρ_s, is

$$\rho_s = s\rho, \tag{2.36}$$

the corresponding density of fresh water is

$$\rho_w = w\rho, \tag{2.37}$$

where $w = m_w/m$ is the mass density of pure water. Obviously, $m = m_s + m_w$, or $s + w = 1$ implies

$$\rho = \rho_s + \rho_w. \tag{2.38}$$

For salt ions exists also a conservation law, their total amount is unchanged when a fluid moves. Similarly like for the total mass, a conservation equation for salt and fresh water can be derived,

$$\frac{\partial \rho_s}{\partial t} = -\boldsymbol{\nabla} \cdot (\mathbf{u_s} \rho_s), \tag{2.39}$$

$$\frac{\partial \rho_w}{\partial t} = -\boldsymbol{\nabla} \cdot (\mathbf{u_w} \rho_w), \tag{2.40}$$

where $\mathbf{u_s}$ is the velocity of the salt ions, $\mathbf{u_w}$ is the mean velocity of the water molecules. Adding both the equations gives

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \rho \cdot (s\mathbf{u_s} + w\mathbf{u_w}). \tag{2.41}$$

The velocity

$$\mathbf{u} = s\mathbf{u_s} + w\mathbf{u_w}, \tag{2.42}$$

describes the center of mass velocity of fresh water and salt. With this

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot (\mathbf{u} \rho). \tag{2.43}$$

However, considering the equation for the salt concentration ρ_s, another important relation can be derived by adding

$$\frac{\partial \rho_s}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{u} \rho_s) = -\boldsymbol{\nabla} \cdot ((\mathbf{u_s} - \mathbf{u})\rho_s), \tag{2.44}$$

$$= -\boldsymbol{\nabla} \cdot \mathbf{J}_s. \tag{2.45}$$

The flux

$$\mathbf{J}_s = -(\mathbf{u_s} - \mathbf{u})\rho_s \tag{2.46}$$

arises from the different velocity of the total mass (salt + water) and the system of salt ions. This difference may stem for example from diffusion and Ficks law may help to find an expression for $\mathbf{J}_s$. Remarkably, this result is derived without any assumptions on the nature of diffusion.

Here we need to make an important statement: There is no diffusion of total mass but there may be diffusion of the constituents of sea water. There are wrong equations introducing mass diffusion in many text books on oceanography!

2.2.3. The general conservation of intensive quantities

Let us now consider an arbitrary intensive quantity, say χ. Intensive means, taking twice of the volume of a fluid, χ remains unchanged. Examples are salt concentration, temperature, nutrient content ...

It may happen that this quantity is not conservative, this means its concentration may change within a fluid element. For example there may be short wave radiation (solar radiation) penetrating from the surface into the water column leading to some heating. Hence, the energy content and the temperature of a fluid cell may change. Another well known example is related to photosynthesis, governed by the chemical formula

$$2CO_2 + 2H_2O + photons \rightarrow 2(CH_2O) + 2O_2 \tag{2.47}$$

There is a sink of dissolved carbon dioxide and a source of oxygen as well as of carbohydrate, which must be considered in the budget equations of those quantities. Moreover, sources and sinks are not independent, their strength is coupled by the stoichiometry of the reactions. Hence, general transport relations for dissolved substances have the form

$$\frac{\partial \rho\chi}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{u}\rho\chi) \ = \ -\boldsymbol{\nabla} \cdot \mathbf{J}_\chi + So_\chi - Si_\chi. \tag{2.48}$$

2.2.4. Dynamic variables

The location $\mathbf{x}$ and the velocity of a fluid elements describe the kinematics of a moving fluid. Likewise, the acceleration as time derivative of the velocity can be used too. However, the dynamic quantity describing the reaction of a body to the action of forces is the momentum. For solid bodies often the total momentum $P = Mv$ is used, where M is the total mass and v the center of mass velocity. Within an inertial system the momentum of a body is conserved as long as no forces are acting on this body. For fluids the concept of generalised densities is more appropriate. Hence, we consider *mass density* (colloquial *density*) $\rho = \frac{\delta M}{\delta V}$, momentum density $\rho\mathbf{u}$, force density, energy density and so on. Unfortunately, the density is not conserved if the fluid is compressed or undergoes thermal expansion. This may make general considerations overly complex and requires sophisticated approximations. We will use two of them frequently:

- the assumption of incompressible flow,

- the Boussinesq[3] approximation

2.2.5. The momentum budget of a fluid element

The momentum of a fluid element, $\rho\mathbf{u}$ (strictly speaking the momentum density within the fluid), is also a non-conservative fluid property. Generally there exists a momentum flux due to advection similarly to dissolved matter density. Additionally, there may be forces acting on a fluid element. There are two types of forces:

[3]Joseph Valentin Boussinesq (* 13. March 1842 in Saint-Andr-de-Sangonis (Dpartement Hrault); †19. February 1929. Lille, Paris.

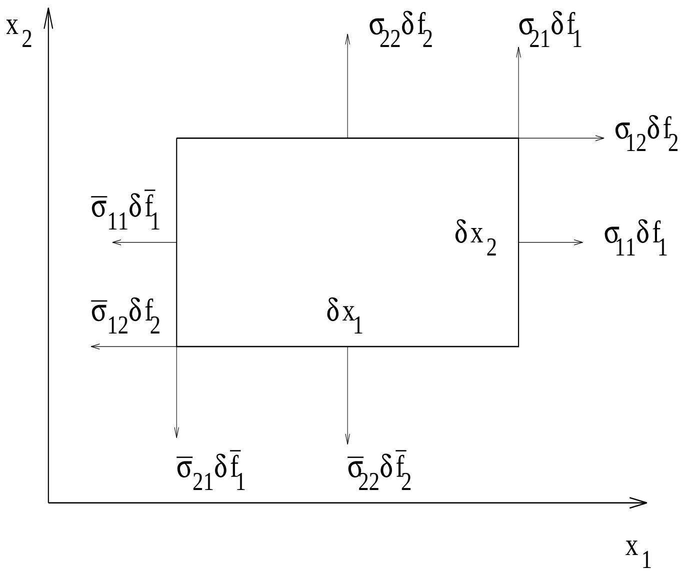


Figure 2.3. Forces acting on the surface of a parallelepipeds.

- long range forces, $\mathbf{f}^l$, like gravity, including inertial forces like centrifugal and Coriolis force,
- short range forces, $\mathbf{f}^s$, acting between the constituents (atoms, molecules) of the fluid. These forces are of electrostatic nature, and can be found using quantum statistical methods. Phenomenological descriptions are for example that of a van der Waals force with a long ranging attractive part and a strong repulsive part for short molecular distances. These forces may act normally or tangentially to the surface of a fluid element.

Figure 2.3 shows some of the contributions, the quantities σ_{ij} describe the force density. The total force acting on a volume element is the integral over its surface, or using Gauss integral theorem, the volume integral over the divergence of the short range forces

$$\mathbf{F} = \oint_A d\mathbf{A} \cdot \sigma = \int dV \boldsymbol{\nabla} \cdot \sigma \tag{2.49}$$

The force density σ can be arranged into a 3-dimensional matrix σ_{ij} with some special conservation properties when it undergoes a co-ordinate transformation. It is a tensor of rank 2 called the *stress tensor*. There exists a proof of Boltzmann[4] that it must be symmetric, otherwise spontaneous rotation of fluid elements would be possible.

[4]Ludwig Eduard Boltzmann, * 20. February 1844 in Wien; †5. September 1906 in Duino near Triest, Graz, Munich, Vienna.

The diagonal elements of σ act perpendicularly to the fluids surface, the off-diagonal elements correspond to tangential stresses. The inward directed normal stress is known as mechanical pressure, the mean pressure is defined by the trace of the stress tensor by

$$p = -\frac{1}{3}\sigma_{ii}. \tag{2.50}$$

In many cases the tangential stress is much smaller than the pressure forces, within an ideal fluid the stress fully vanishes. Hence, it is useful to separate pressure and tangential stress,

$$\sigma_{ij} = -p\delta_{ij} + \Sigma_{ij}. \tag{2.51}$$

(Ocean water is isotropic and we have $p_i = p$.) The remaining tensor Σ_{ij} corresponds to friction and is called *friction tensor*.

We summarize our momentum budget,

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot \rho \mathbf{u}\mathbf{u} = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \boldsymbol{\Sigma} + \mathbf{f}^l, \tag{2.52}$$

and find, that the momentum change of a fluid element is driven by three forces:

- gradients of the pressure,
- divergence of the friction tensor,
- long range forces.

2.2.6. Friction and mean flow

The momentum budget Eq. 2.52 has the pressure gradient and the elements of the friction tensor as unknown quantities at the right hand side. A simple experiment helps to express the elements of Σ is terms of the flow velocity. First, a linear velocity profile as shown in Figure 2.4 develops, when lid is sliding over the surface of a fluid exerting a tangent stress σ on the fluid. Second, the stress is proportional to the lids velocity U and inversely proportional to the distance of the lids h. Hence, the velocity profile is

$$u(z) = U\frac{z}{h}, \tag{2.53}$$

and the stress is

$$\sigma = \eta\frac{U}{h} = \eta\frac{\partial u}{\partial z}. \tag{2.54}$$

The viscosity η is a material property of the fluid. Since the water is not accelerating, this stress must be constant through the water column and acts also on the lower lid. Here it is balanced by an opposite stress from below, keeping the whole experimental setup at its position. Hence, x-directed momentum is propagating downward through the

Figure 2.4. Velocity profile from friction within a Newtonian fluid.

fluid and is leaving the fluid through the lower lid. This leads us to another important physical meaning of the stress Σ_{ij}, it describes the propagation of j-directed momentum into direction i, it describes a *momentum flux*. In our example, 1 denotes the x-direction and 3 denotes the z-direction, we have

$$\Sigma_{13} \sim \eta \frac{\partial u}{\partial z}. \tag{2.55}$$

This finding suggests, that the friction tensor Σ is related to the deformation tensor,

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2.56}$$

introduced in the kinematics section. It is symmetric as required but its trace corresponds to the flow divergence, hence to the compression of the fluid element,

$$D_{ii} = \boldsymbol{\nabla} \cdot \mathbf{u} = \frac{1}{V}\frac{dV}{dt}. \tag{2.57}$$

The friction tensor is not related to compression and so the most simple relation between the friction tensor and the deformation tensor should have the form,

$$\Sigma_{ij} = 2\eta\left(D_{ij} - \frac{1}{3}D_{ll}\delta_{ij}\right). \tag{2.58}$$

Applying the divergence operator to Σ_{ij} yields our final equation for the momentum budget

$$\begin{aligned} \frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j u_i) &= -\frac{\partial p}{\partial x_i} + \frac{\partial \Sigma_{ji}}{\partial x_j} + {f_i}^l, \\ &= -\frac{\partial p}{\partial x_i} + \eta\left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{3}\frac{\partial^2 u_j}{\partial x_i \partial x_j}\right) + {f_i}^l, \end{aligned} \tag{2.59}$$

or equivalently

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \mathbf{u} + \frac{\eta}{3}\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) + \mathbf{f}^l, \tag{2.60}$$

This equation is called the Navier-Stokes equation. It is the basic equation for many applications of hydrodynamics. It has two major ingredients, 1) the exact momentum budget, hence the acceleration of fluid elements by forces (generalization of Newton's second law of mechanics for fluids), and 2) a much weaker founded relation between the friction tensor and the shear of the flow including the definition of the viscosity η as a material property. At the one hand side this is justified by experimental verification (about 2 centuries of applied hydrodynamics) but also from a more rigorous derivation from "first principles" (quantum statistics using Schrödingers equation, the second law of thermodynamics and the atom-atom interaction as the basic force).

The Navier-Stokes equations are not complete. They are second order partial differential equations and require boundary conditions for its solution. These will be specified later. There are also two remaining unknown quantities, the density ρ and the pressure p. To find a method to define them, we need to consider some thermodynamics.

Exercise 2.4 Assume an irrotational flow ($\boldsymbol{\nabla}\times\mathbf{u} = 0$) and derive the Bernoulli equation! This requires the expression of the flow as gradient of a scalar potential.

2.2.7. Coriolis and centrifugal force

We are interested in fluids moving on a planet that itself is moving in a complex way. There are two major types of long range external forces acting on a fluid element from outside the fluid,

- gravitation of the earth, the sun and the moon. The forces can be expressed as gradient of a gravitation potential. The gravitation of the earth is mostly considered to be constant, the influence of sun and moon depend on their relative position to the earth and varies with time. A more accurate consideration reveals also internal movements of the earth's core driven by moon, sun and rotation of the earth. Here we will not consider this effect, it may be relevant for exact determination of the sea level with tidal models.

- centrifugal and Coriolis force as result of the earth's rotation. Although tidal forces play the major role at daily time scales, they cause a mostly oscillating ocean motion. Hence, centrifugal and Coriolis forces are of major interest if wind driven currents and thermohaline circulation is considered.

The Coriolis force is named after the French scientist Gaspard-Gustave de Coriolis[5] who published the first theory in the frame of theoretical mechanics. Earlier, the existence of this force was notice in the field of ballistics (eastward deflection of a bullet fired northward), meteorology (rule of Buys Ballot[6] for the relation between air pressure gradients and the wind direction), theory of tides (Laplace[7]). William Ferrel[8] used the concept to understand the existence of atmospheric circulation cells with prevailing west-ward directed winds (1856). The best know example should be Foucault's pendulum experiment carried out 1851 to demonstrate the rotation of the earth. In the beginning of the 20th century oceanographers and meteorologists were fully aware on the importance of the Coriolis force. Nevertheless, there was an endless controversial discussion on the physical background - at least some justification was wrong from a physical point of view. So it was no outdated to publish a book "An introduction to the Coriolis force" by Henry Stommel[9] (1989).

Understanding of the Coriolis force is not simple. It is often introduced as a fictitious force, but for sure it has real effects. There is an interesting article by Anders Persson[10][11] on understanding the Coriolis forces, which should be strongly recommended for students. He writes on the famous approach of Henry Stommel "The late Henry Stommel appreciated the sense of frustration that overcomes students in meteorology and oceanography who encounters the "mysterious" Coriolis force as a result of a series of "formal manipulations": *Clutching the teacher's hand, they are carefully guided across a narrow gangplank over the yawning gap between the resting frame and the uniformly rotating frame. Fearful of looking down into the cold black water between the dock and the ship, many are glad, once safely aboard, to accept the idea of a Coriolis force, more or less with blind faith, confident that it has be derived rigorously. And some people prefer never to look over the side again.* (Stommel and Moore, 1989. *An introduction to the Coriolis force.*).

In this lesson on theoretical oceanography we go just this way over the "gangplank". But it is suggested, to look over the side several times! The detailed "Gedankenexperiment"

[5]Gaspard Gustave de Coriolis, * 21. May 1792 in Paris; †19. September 1843 in Paris. Paris.

[6]Christoph Heinrich Dietrich Buijs Ballot,* 10. October 1817 in Kloetinge, (Provinz Zeeland), †. February 1890 in Utrecht. Kloetinge, Utrecht.

[7]Pierre-Simon (Marquis de) Laplace, * 28. March 1749 in Beaumont-en-Auge (Normandie), †5. March 1827 in Paris. Caen, Paris

[8]William Ferrel, * 29. January 1817 in Bedford-, (Pennsylvania, USA), †18. September 1891 in Martinsburg (West Virginia). Nashvill, Cambridge (Mas).

[9]Henry Melson Stommel, * 27. September 1920 in Wilmington, (Delaware), †17. Januar 1992. Yale, Woods Hole, Havard, MIT

[10]http://onlinelibrary.wiley.com/doi/10.1002/j.1477-8696.2000.tb04052.x/pdf

[11]http://www.science.unitn.it/∼ fisica1/fisica1/appunti/mecc/appunti/cinematica/Coriolis_persson.pdf

described in the textbook of Apel may help to achieve a deeper understanding.

We start with the momentum equations within a inertial system, like those derived previously and rewrite them in a coordinate frame fixed at the earth, hence, rotating with the earth and revolving round the sun together with the earth, like that of an observer living at a fixed point at the earth surface. Consider the movement of a vector pointing

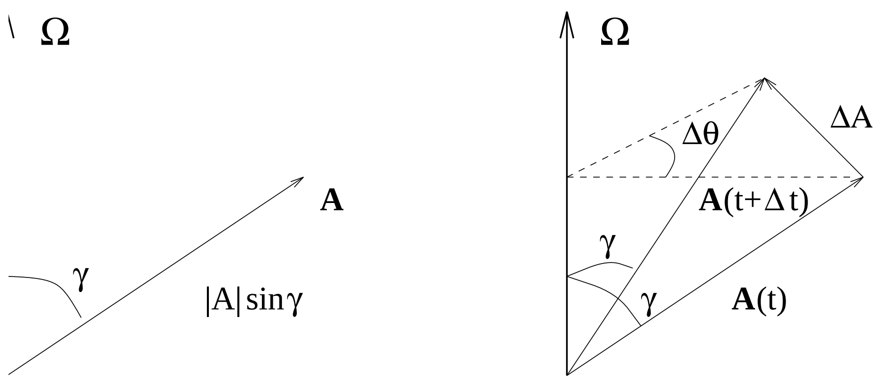


Figure 2.5. Rotation of a vector **A** with an angular velocity **Ω**.

from the earth's center of gravity to an arbitrary fixed point somewhere on the earth. Within an inertial system, the vector is changing in time, (see Fig. 2.5

$$\Delta\mathbf{A} \;=\; \mathbf{A}(t+\Delta t) - \mathbf{A}(t) = \mathbf{n}|\mathbf{A}|\sin\gamma\Delta\theta + \mathcal{O}((\Delta\theta)^2). \qquad (2.61)$$

n is a unity vector pointing into the current direction of the vector's movement. This vector is perpendicularly to both, **A** and**Ω**,

$$\mathbf{n} \;=\; \frac{\boldsymbol{\Omega}\times\mathbf{A}}{|\boldsymbol{\Omega}\times\mathbf{A}|}. \qquad (2.62)$$

In the infinitesimal limit, $\Delta t \to 0$, the velocity of **A** is,

$$\lim_{\Delta t\to 0}\frac{\Delta\mathbf{A}}{\Delta t} \;=\; \frac{d\mathbf{A}}{dt} = |\mathbf{A}|\sin\gamma\frac{d\theta}{dt}\frac{\boldsymbol{\Omega}\times\mathbf{A}}{|\boldsymbol{\Omega}\times\mathbf{A}|}. \qquad (2.63)$$

Using the identities

$$|\boldsymbol{\Omega}\times\mathbf{A}| \;=\; |\boldsymbol{\Omega}||\mathbf{A}|\sin\gamma,$$
$$|\boldsymbol{\Omega}| \;=\; \frac{d\theta}{dt} \qquad (2.64)$$

we find the simple expression,

$$\frac{d\mathbf{A}}{dt} = \mathbf{\Omega} \times \mathbf{A}. \tag{2.65}$$

The changing of **A** is visible in the inertial system, but not in the system fixed at the earth.

Now we consider a vector, **B**, that changes with time also with respect to the system fixed at the earth. Its coordinate representation is

$$\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}, \tag{2.66}$$

i, **j** and **k** are the unity vectors in the rotating system. The time derivative of **B** in the rotating coordinate system is a time derivative of the components, but not of the unit vectors,

$$\left(\frac{d\mathbf{B}}{dt}\right)_r = \frac{dB_1}{dt}\mathbf{i} + \frac{dB_2}{dt}\mathbf{j} + \frac{dB_3}{dt}\mathbf{k}. \tag{2.67}$$

The index r denotes the derivative in the rotating system. In the inertial system both, the components as well as the unit vectors are changing:

$$\left(\frac{d\mathbf{B}}{dt}\right)_i = \frac{dB_1}{dt}\mathbf{i} + \frac{dB_2}{dt}\mathbf{j} + \frac{dB_3}{dt}\mathbf{k} + B_1\frac{d\mathbf{i}}{dt} + B_2\frac{d\mathbf{j}}{dt} + B_3\frac{d\mathbf{k}}{dt}. \tag{2.68}$$

The unity vectors are special instances of the more general aforementioned vector **A**. Hence, the time derivative of the vector components within the rotating frame is supplemented by the movement of the unity vectors with the rotating earth,

$$\left(\frac{d\mathbf{B}}{dt}\right)_i = \left(\frac{d\mathbf{B}}{dt}\right)_r + \mathbf{\Omega} \times \mathbf{B}. \tag{2.69}$$

Applying this to the position vector we find the transformation rule between the velocity in the inertial and the rotating system,

$$\left(\frac{d\mathbf{r}}{dt}\right)_i = \left(\frac{d\mathbf{r}}{dt}\right)_r + \mathbf{\Omega} \times \mathbf{r}, \tag{2.70}$$

and for the velocity

$$\mathbf{u}_i = \mathbf{u}_r + \mathbf{\Omega} \times \mathbf{r}. \tag{2.71}$$

This is nothing else than the simple statement that velocity in the inertial frame consists of the two components "velocity within the rotating system" and "rigid rotation of the rotating system".

Applying this transformation to the velocity in the inertial frame we find similarly

$$\left(\frac{d\mathbf{u}_i}{dt}\right)_i = \left(\frac{d\mathbf{u}_i}{dt}\right)_r + \mathbf{\Omega} \times \mathbf{u}_i. \tag{2.72}$$

Now we substitute $\mathbf{u}_i$ at the right hand side by $\mathbf{u}_r$,

$$\left(\frac{d\mathbf{u}_i}{dt}\right)_i = \left(\frac{d\mathbf{u}_r + \boldsymbol{\Omega}\times\mathbf{r}}{dt}\right)_r + \boldsymbol{\Omega}\times(\mathbf{u}_r + \boldsymbol{\Omega}\times\mathbf{r})$$
$$= \left(\frac{d\mathbf{u}_r}{dt}\right)_r + 2\boldsymbol{\Omega}\times\mathbf{u}_r + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}). \quad (2.73)$$

Hence, the acceleration in the rotating system consists of the acceleration in the inertial frame driven by the external (gravity forces) and is complemented by two additional forces acting only if the fluid parcel is fixed to the rotating frame:

- the Coriolis force $-2\rho\boldsymbol{\Omega}\times\mathbf{u}_r$, perpendicularly to both, $\boldsymbol{\Omega}$ and $\mathbf{u}_r$.
- the centrifugal force $-2\rho\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$, acting in the direction of the component of $\mathbf{r}$ that is perpendicularly to the axis of rotation, $\mathbf{r}_\perp$.

For the centrifugal force there exists a also a representation as gradient of a potential function. To derive this we decompose the position vector into components parallel and perpendicular to $\boldsymbol{\Omega}$, $\mathbf{r} = \mathbf{r}_\parallel + \mathbf{r}_\perp$. The vector product $\boldsymbol{\Omega}\times\mathbf{r}_\parallel$ vanishes and with $|\mathbf{r}_\perp| = |\mathbf{r}|\cos\varphi$ we end up with the potential representation

$$\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r}) = -\Omega^2\mathbf{r}_\perp = -\frac{1}{2}\boldsymbol{\nabla}\left(\Omega^2 r^2\cos^2\varphi\right). \quad (2.74)$$

φ is the geographical latitude. Hence, the centrifugal force grows with the distance from the axis of rotation. Following the surface of the earth, it has maximum amplitude at the equator an vanishes near the poles.
Exercise 2.5 Verify equation 2.74!
Exercise 2.6 In which direction points the vector of the Coriolis force?

2.2.8. The momentum equations on the rotating earth

Friction, pressure gradients as well as any thermodynamic property must not depend an the coordinate system. Hence, the formulation of the Navier-Stokes equation for a fluid in a rotating system is simple, (the subscript r is dropped)

$$\frac{\partial\rho\mathbf{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}\mathbf{u}) = -2\rho\boldsymbol{\Omega}\times\mathbf{u} - \boldsymbol{\nabla}p + \boldsymbol{\nabla}\boldsymbol{\Sigma} - \rho\boldsymbol{\nabla}\Phi + \mathbf{f}^{tide}, \quad (2.75)$$

The potential Φ is the sum of the gravitational potential and the potential of the centrifugal force,

$$\Phi = \Phi_E + \frac{1}{2}\Omega^2 r^2\cos^2\varphi. \quad (2.76)$$

It is called the effective gravitation potential. It can be shown that for an ocean at rest the sea surface is an equipotential surface of Φ. The planet is deformed to-wards the shape of a rotational ellipsoid but did not fully develop this shape. The earth has a more irregular form because of the inhomogeneous mass distribution in the earth's mantle and crust. An ellipsoid is a good approximation for our purposes. The gravitation potential can be written as a power series of Legendre polynomials,

$$\Phi_E(r\varphi\lambda) \approx -\gamma\frac{M_E}{r}\left(1-\delta\frac{r_E^2}{r^2}\left(3\sin^2\varphi-1\right)+\epsilon(r\varphi\lambda)\right). \tag{2.77}$$

Here $r_E = 6367.5$ km is the average radius of the earth, $\delta = 0.54 \cdot 10^{-3}$ is the oblateness of the earth. ϵ describes perturbations from inhomogeneous mass distributions and deviations from the rotation ellipsoid. It is small, $\epsilon \approx 10^{-6}$, but must be considered for navigation (GPS), satellite altimetry and other applications where the exact position on the earth is needed.

Hence, the effective gravitation force is directed perpendicularly to the equipotential surface. This defines upward" and "downward" in a local coordinate system. Moreover, approximating the earth as a sphere, the effective gravitation may vary meridional, but the gradient must be directed to the center of the earth.

2.2.9. Approximations for the Coriolis force

The expression for the Coriolis force

$$F_c = -2\rho\mathbf{\Omega}\times\mathbf{u} \tag{2.78}$$

is linear but depends non-linearly on the coordinates. Working in spherical coordinates is complicated searching for analytical solutions of the hydrodynamic equations. Some common simplifications may be of great help and will be introduced below.

The rotation of the earth must be counted in an inertial system. It is not a solar day, but good approximation is the system of "fixed" stars, where Ω is

$$\begin{aligned}\Omega &\approx 2\pi\left(1+\frac{1}{365.24}\right) \quad \mathrm{rad\,d^{-1}} \\ &\approx 7.292\cdot10^{-5} \quad \mathrm{rad\,s^{-1}}.\end{aligned} \tag{2.79}$$

The f-plane approximation

The expression for the Coriolis force in the local tangent plane reads

$$-2\rho\mathbf{\Omega}\times\mathbf{u} = -2\rho\Omega\left[i\left(w\sin\theta - v\cos\theta\right) + ju\cos\theta - ku\cos\theta\right] \tag{2.80}$$

(u, v, w) is the tangent velocity vector, (i, j, k) the unity vectors and θ the angle counted from the north pole. The geographical latitude is $\lambda = 90° - \theta$.

Remarkly, this expression is not symmetric. Vertical motion on the earth results always in a zonal deflection. This contribution is largest at the equator and vanishs at the poles. However, vertical velocity is usually much smaller than the horizpontal velocity component and this contribution may be neglected.

The Coriolis force has also an vertical component resulting from the zonal velocity. This contribution is small at the equator and largest at the poles. It is usually not considered, since it is much smaller than gravity and appears as a minor correction to the effective gravity **g**.

Hence, only two contributions of the Coriolis force are of importance here,

$$-2\rho\mathbf{\Omega} \times \mathbf{u} \approx -2\rho\Omega\cos\theta\,(-iv + ju)\,. \tag{2.81}$$

The next important approximation is to consider small horizontal elevations of fluid particles. In this case the angle θ is constant and the Coriolois force is described by the Coriolis parameter

$$f = 2\Omega\cos\theta = 2\Omega\sin\lambda. \tag{2.82}$$

f is also called *planetary vorticity.*

The β-plane approximation

Excursions of water parcels over several latitudes are possible. In this case the variability of f becomes important. We will see later that a special wave type, the *planetary Rossby waves*, is related to meridional variations of the Coriolis force. To account for this variability, a local Taylor expansion is empoyed,

$$f = f_0 + \frac{\partial f}{\partial y} y = f_0 + \beta y, \tag{2.83}$$

$$\beta = \frac{2\Omega}{R_e}\cos\lambda_0. \tag{2.84}$$

Exercise 2.7 Find f and β at the poles, for the Baltic Sea, at low latitudes (10°N/S) and at the equator.

2.3. Thermodynamics

From a drop of water a logician could predict an Atlantic or a Niagara. Sir Arthur Conan Doyle (1859 - 1930). Sherlock Holmes in "A study in Scarlet".

Thermodynamics deal with energy considerations for continuous bodies like gasses, fluids or solids. The considerations used here are based on the concept of (near) *equilibrium* motion in relation to *conservation* of matter and energy. In addition *irreversibly* has to be considered expressed in terms of a maximum principle for the state variable entropy.

2.3.1. Extensive and intensive quantities

In the following considerations all quantities are weighted with the mass of a fluid element. Let M be its total mass and M_i the masses of its constituents, say salt and fresh water, we find for the weighted masses $m_i = M_i/M$

$$\sum_i m_i = 1, \quad \sum_i dm_i = 0, \quad \sum_i \rho_i = \rho. \tag{2.85}$$

For the specific volume $v = 1/\rho$ we find for sea water

$$v = m_s v_s + m_w v_w \qquad \text{with} \quad v_s = \frac{\partial v}{\partial m_s}, v_w = \frac{\partial v}{\partial m_w}. \tag{2.86}$$

Note, that this approach is only valid, if the intensive quantities are independent of the total mass M. This may be violated, if surface effects (surface tension) becomes important. **Exercise 2.8** Proof the above relation! Start from the fact, that the volume as function of the partial masses, pressure and temperature must double, when all all partial masses are doubled: $V(aM_s, aM_w, p, T) = aV(M_s, M_w, p, T)$. Differentiate both sides to a and consider the case $a = 1$! Verify

$$V = \frac{\partial V}{\partial M_s} M_s + \frac{\partial V}{\partial M_w} M_w! \tag{2.87}$$

Now consider $a = 1/M$ and verify

$$v(m_s, m_w, p, T) = \sum_i v_i m_i \qquad \text{with} \quad v_i = \frac{\partial v}{\partial m_i}! \tag{2.88}$$

Is v an intensive or extensive variable?

2.3.2. Traditional approach to sea water density

The traditional approach to sea water thermodynamics is to measure density, compressibility or specific heat with high accuracy as function of temperature, pressure and salinity and find a polynomial approximation

$$\rho(S, T, p) = \rho_0 + \sum_{ijk} a_{ijk} S^i T^j p^k \tag{2.89}$$

Relative accuracy up 10^{-5} is required. An international standard was set of the so called UNESCO-formulas, complex polynomial approximations for various thermodynamic quantities. Each single formula has sufficient accuracy, however, the relations between many quantities (Maxwell relations, see below) are not fulfilled. For this reason, the UNESCO-formula has been replaced recently by the so called TEOS-10 standard approach based on a consistent approach using thermodynamic potentials. This will be introduced shortly below.

For some special investigations, e.g. analytical calculation where less accuracy is needed, simple approximations may by of some help. If temperature and salinity vary only slightly around T_0 and S_0 approximations of the form

$$\rho = \rho_0 \left((1 - \alpha_T (T - T_0)) - \alpha_s (S - S_0) \right) \tag{2.90}$$

may be useful. A more sophisticated formula is this of Ekart (1958),

$$
\begin{aligned}
\rho &= \frac{P_0}{\lambda + \alpha_0 P_0}, \\
\lambda &= 1779.5 + 11.25T - 0.775T^2 - (3.8 + 0.1T)S, \\
\alpha_0 &= 0.698, \\
P_0 &= 5890 + 38T - 0.375T^2 + 3S. \qquad (2.91)
\end{aligned}
$$

Other equations consider also the influence of suspended matter. This is needed for example in models of so called turbidity currents.

2.3.3. Mechanical, thermal and chemical equilibrium

A thermodynamic system, for example a fluid element may interact with its environment in different ways. This interaction can bring for example two subsystems into some equilibrium:

- mechanical equilibrium. At rest (no friction) all forces are balanced, the pressure of both subsystems is the same.

- thermal equilibrium. Heat is exchanged until the temperature of both systems is the same.

- chemical equilibrium. Substances are exchanged (for example by diffusion) until the chemical potentials are the same.

2.3.4. First law of thermodynamics

The first law of thermodynamics considers the energy of a subsystem. Its internal energy density, E, (kinetic energy per mass of the molecules, chemical energy of dissolved substances like solution energy of salt) may be changed by mechanical work and by heat exchange as well as by the exchange of matter,

$$dE = -pdv + \delta Q + \sum_i H_i dm_i. \qquad (2.92)$$

p is the pressure, dv the specific volume change, δQ the exchanged heat, H_i the enthalpy density of substance i (for example H_s and H_w for salt and fresh water) and dm_i the exchanged mass densities. (Q is note a state variable, hence δQ is not a differential.)

2.3.5. Second law of thermodynamics

A suitable form of the second law of thermodynamics is the statement, that there exists a state variable entropy, η, that is changed by heat exchange δQ like

$$d\eta = \frac{\delta Q}{T} + \sum_i \eta_i dm_i. \qquad (2.93)$$

This defines also the absolute temperature T and the specific entropy of a constituent η_i. Here i refers again either to salt or fresh water. A second statement made in the second law of thermodynamics is the fact, that Eq. (2.93) is the lower limit for the entropy change, if the heat exchange is irreversible, the entropy change is always greater than this value.

2.3.6. Thermodynamics potentials

Eliminating the heat exchange δQ the Gibbs equation can be derived from the first and second law of thermodynamics:

$$Td\eta = dE + pdv - \sum_i \mu_i dm_i, \tag{2.94}$$

where the chemical potential of constituent i is introduced,

$$\mu_i = H_i - T\eta_i \tag{2.95}$$

Entropy, internal energy, specific volume and mass of constituents are extensive variables. Using the Euler theorem, it can be shown, that in this case also the following general relation holds,

$$T\eta = E + pv - \sum_i \mu_i m_i, \tag{2.96}$$

This is the key to define *thermodynamical potentials*

- internal energy $E(\eta, T, m_i)$,
- free energy $F(T, v, m_i) = E - T\eta$,
- enthalpy $H(\eta, p, m_i) = E + pv$,
- free enthalpy $G(T, p, m_i) = E + pv - T\eta = \sum_i \mu_i m_i$.

If these quantities are given in its "natural" variables, total differentials can be derived suitable to calculate all other thermodynamic variables.

$$dE = \sum_i \mu_i dm_i - pdV + Td\eta, \tag{2.97}$$

$$dF = \sum_i \mu_i dm_i - pdV - \eta dT, \tag{2.98}$$

$$dH = \sum_i \mu_i dm_i + Td\eta + vdp, \tag{2.99}$$

$$dG = \sum_i \mu_i dm_i - \eta dT + vdp. \tag{2.100}$$

In oceanography pressure and temperature are simple to measure, entropy or density are not. Hence, the free enthalpy (or Gibbs potential) is the best choice for a thermodynamical potential. Since we have only two constituents, water and salt, the chemical potential can be simplified with

$$\sum_i \mu_i dm_i = \mu_s dS + \mu_w d(1-S) = \mu dS. \tag{2.101}$$

$$dG = \mu dS - \eta dT + v dp. \tag{2.102}$$

Derived state variables are,

$$\mu = \left(\frac{\partial G}{\partial S}\right)_{pT}, \tag{2.103}$$

$$\eta = -\left(\frac{\partial G}{\partial T}\right)_{pS}, \tag{2.104}$$

$$v = \left(\frac{\partial G}{\partial p}\right)_{TS} = \frac{1}{\rho}. \tag{2.105}$$

Other quantities of basic interest are the thermal expansion coefficient α, haline contraction, γ, isothermal compressibility, κ, or specific heat, c_p,

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right) = \left(\frac{\partial^2 G}{\partial T \partial p}\right) / \left(\frac{\partial G}{\partial p}\right), \tag{2.106}$$

$$\gamma = -\frac{1}{v}\left(\frac{\partial v}{\partial S}\right) = \left(\frac{\partial^2 G}{\partial S \partial p}\right) / \left(\frac{\partial G}{\partial p}\right), \tag{2.107}$$

$$\kappa = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right) = \left(\frac{\partial^2 G}{\partial p^2}\right) / \left(\frac{\partial G}{\partial p}\right), \tag{2.108}$$

$$c_p = -T\left(\frac{\partial^2 G}{\partial T^2}\right). \tag{2.109}$$

Very useful are the so called Maxwell relations, derived from the second order derivatives,

$$\frac{\partial \eta}{\partial S} = -\frac{\partial \mu}{\partial T}, \qquad \frac{\partial \mu}{\partial p} = \frac{\partial v}{\partial S}, \qquad \frac{\partial v}{\partial T} = -\frac{\partial \eta}{\partial p}. \tag{2.110}$$

The estimation of the free enthalpy is a task of precision measurements of temperature, salinity, density, sound velocity, freezing temperature, specific heat, evaporation heat, triple point temperature. This task has been completed only recently with a sufficient accuracy. The use of the free enthalpy is not common among oceanographers yet but is international standard since 2010.

2.4. Energy considerations

Here we consider the energy changes of a fluid element. The aim is to derive a advection-diffusion equation for the temperature, which is an essential tool for any ocean or atmosphere models. The total energy of a fluid parcel can be changed by

- by external fluxes of heat or radiation
- mechanical work on its surface
- mechanical work on the volume forces acting on the fluid parcel
- exchange of chemical energy by particle fluxes.

The total energy is the sum of the potential and the kinetic energy. Hence, its time derivative has the general form

$$\rho \frac{D}{Dt}(E+E_k) \;=\; -\boldsymbol{\nabla}\cdot\mathbf{J}^{tot} - \rho\mathbf{u}\cdot\boldsymbol{\nabla}\Phi. \tag{2.111}$$

The time tendency of the kinetic energy can be derived from the momentum equation, Eq. 2.75, (the tidal potential is part of Φ for the moment)

$$\frac{\partial \rho\mathbf{u}}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}\mathbf{u}) \;=\; -2\rho\boldsymbol{\Omega}\times\mathbf{u} - \boldsymbol{\nabla}p + \boldsymbol{\nabla}\boldsymbol{\Sigma} - \rho\boldsymbol{\nabla}\Phi \tag{2.112}$$

by multiplication with $\mathbf{u}$,

$$\rho\frac{D}{Dt}(E_k) \;=\; -\mathbf{u}\cdot\boldsymbol{\nabla}p + \mathbf{u}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\Sigma}) - \rho\mathbf{u}\cdot\boldsymbol{\nabla}\Phi. \tag{2.113}$$

Exercise 2.9 Verify the expression for time derivative of the kinetic energy! Show that the work (change of kinetic energy) due to the Coriolis force vanishes! Discuss this result!

From the product rule we find,

$$\mathbf{u}\cdot\boldsymbol{\nabla}\sigma \;=\; \boldsymbol{\nabla}\cdot(\mathbf{u}\sigma - (\sigma\cdot\boldsymbol{\nabla})\,\mathbf{u}.) \tag{2.114}$$

The second term is related to the deformation tensor $D_{ij} = \partial u_i/\partial x_j$,

$$(\sigma\cdot\boldsymbol{\nabla})\,\mathbf{u} \;=\; (-p\delta_{ij} + \Sigma_{ij})\,D_{ij}. \tag{2.115}$$

The second term describes the work done by shear forces usually called "friction" or from the energetic aspect "dissipation", and we define dissipation as

$$\rho\epsilon \;=\; \Sigma_{ij}D_{ij}. \tag{2.116}$$

(Note, the diagonal elements of D corresponding to compression do not contribute, since the trace of Σ vanishes.)

The effective gravitation potential is constant in time, but the tidal potential varies, since the relative position of moon and sun is changing. Hence, we get

$$\frac{D\Phi}{Dt} = \frac{\partial \Phi^{tide}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\Phi. \tag{2.117}$$

for the total mechanical energy

$$\rho \frac{D}{Dt}\left(E_k + \Psi\right) = -\boldsymbol{\nabla}\left(-\mathbf{u} \cdot \sigma\right) + p\boldsymbol{\nabla}\mathbf{u} - \rho\epsilon + \rho\frac{\partial \Phi^{tide}}{\partial t}. \tag{2.118}$$

The total mechanical energy changes due to a divergence in the flux of mechanical work $-\mathbf{u} \cdot \sigma$, compression, dissipation and a time dependent external gravity potential.

We return to the time tendency of the total energy

$$\rho \frac{D}{Dt}\left(E + E_k\right) = -\boldsymbol{\nabla} \cdot \mathbf{J}^{tot} - \rho\mathbf{u} \cdot \boldsymbol{\nabla}\Phi. \tag{2.119}$$

Subtracting the kinetic energy budget gives the remaining terms for changes of the internal energy,

$$\rho \frac{DE}{Dt} = -\boldsymbol{\nabla} \cdot \mathbf{J}_H - \rho\boldsymbol{\nabla} \cdot \mathbf{u} + \rho\epsilon. \tag{2.120}$$

$\mathbf{J}_H$ is the enthalpy flux summarizing the radiative energy flux, the molecular heat flux, flux of chemical energy,

$$\mathbf{J}_H = \mathbf{J}_{rad} + \mathbf{J}_{heat} + \mathbf{J}_{chem}. \tag{2.121}$$

Note, the loss terms of kinetic energy are sources of internal energy. The flux of mechanical work changes mechanical energy, but not the internal energy.

Especially for discussions of the thermal energy, the enthalpy is more appropriate. It is defined as

$$H = E + pv, \qquad v = \rho^{-1}. \tag{2.122}$$

Using the continuity equation the budget equation for H reads,

$$\rho \frac{DH}{Dt} = \frac{Dp}{Dt} - \boldsymbol{\nabla} \cdot \mathbf{J}_H + \rho\epsilon. \tag{2.123}$$

2.5. Temperature equations

Measuring temperature is easy. Hence, temperature is an important variable in oceanography. Also the equation of state has temperature as one important variable. As we will see later, density determines the pressure and the pressure gradients drive flow. In short, temperature is an important variable and its time tendency due to material fluxes or energy fluxes is of great interest.

2.5.1. The in-situ temperature

The temperature of two fluid elements is the same when energy exchange is in equilibrium. Hence, we start with the enthalpy equation and expand the total differential

$$\rho\left(\left(\frac{\partial H}{\partial S}\right)_{pT}\frac{DS}{Dt}+\left(\frac{\partial H}{\partial T}\right)_{pS}\frac{DT}{Dt}+\left(\frac{\partial H}{\partial p}\right)_{ST}\frac{Dp}{Dt}\right) = \frac{Dp}{Dt}-\boldsymbol{\nabla}\cdot\mathbf{J}_H+\rho\epsilon \quad (2.124)$$

The derivatives are related to material properties,

$$\left(\frac{\partial H}{\partial T}\right)_{pS} = c_p; \qquad \left(\frac{\partial H}{\partial p}\right)_{ST}=\rho^{-1}\left(1-\alpha T\right). \quad (2.125)$$

Hence, the time tendency for the temperature reads

$$\rho c_p\frac{DT}{Dt} = \alpha T\frac{Dp}{Dt}-\boldsymbol{\nabla}\cdot\mathbf{J}_H+Q_T. \quad (2.126)$$

The source term collects dissipative heating and solubility effects,

$$Q_T = \rho\left(\frac{\partial H}{\partial S}\right)_{pT}\boldsymbol{\nabla}\cdot\mathbf{J}_S. \quad (2.127)$$

This equation is not very useful for practical applications. Temperature is not conserved even if the heat flux divergence is small because pressure changes may cause temperature changes.

2.5.2. The conservative temperature

Considering the enthalpy differential in terms of its canonical variables η, p and S we find

$$\rho\left(\left(\frac{\partial H}{\partial S}\right)_{p\eta}\frac{DS}{Dt}+\left(\frac{\partial H}{\partial \eta}\right)_{pS}\frac{D\eta}{Dt}+\left(\frac{\partial H}{\partial p}\right)_{S\eta}\frac{Dp}{Dt}\right) = \frac{Dp}{Dt}-\boldsymbol{\nabla}\cdot\mathbf{J}_H+\rho\epsilon. \quad (2.128)$$

With the Maxwell relation for H,

$$\left(\frac{\partial H}{\partial \eta}\right)_{pS} = T, \qquad \left(\frac{\partial H}{\partial p}\right)_{S\eta}=\rho^{-1}, \quad (2.129)$$

the pressure term cancels out exactly,

$$\rho\left(\left(\frac{\partial H}{\partial S}\right)_{p\eta}\frac{DS}{Dt}T\frac{D\eta}{Dt}\right) = -\boldsymbol{\nabla}\cdot\mathbf{J}_H+\rho\epsilon. \quad (2.130)$$

This is an equation for the entropy. More useful is the definition of the so called *potential enthalpy*, H^0, the enthalpy a water parcel would have when brought adiabatically to the sea surface

$$H^0(S,\eta) = H(S,\eta,p^0). \quad (2.131)$$

Notably, the temperature changes in this case, H^0 does not. This conservative property can be used to find a state variable more convenient for practical purposes.

$$\rho \frac{DH^0}{Dt} = \rho \left(\left(\frac{\partial H^0}{\partial S} \right)_{p\eta} \frac{DS}{Dt} + \left(\frac{\partial H^0}{\partial \eta} \right)_{pS} \frac{D\eta}{Dt} \right) \tag{2.132}$$

We define the *potential temperature*, θ, hence, the temperature a fluid element would have, when brought adiabatically to the surface,

$$\left(\frac{\partial H^0}{\partial \eta} \right)_{pS} = \theta. \tag{2.133}$$

Using the transport equation for salt and the budgets for H and H^0 to eliminate the time derivativ of the entropy, we find

$$\rho \frac{DH^0}{Dt} = \left(\frac{\theta}{T} \left(\frac{\partial H}{\partial S} \right)_{p\eta} - \left(\frac{\partial H^0}{\partial S} \right)_{p\eta} \right) \mathbf{\nabla} \cdot \mathbf{J}_S + \frac{\theta}{T} \left(-\mathbf{\nabla} \mathbf{J}_H + \rho \epsilon \right). \tag{2.134}$$

Now we use a special approximation for the specific heat,

$$c_p^* = 3.991.868 \, \mathrm{J\ kg^{-1} K^{-1}} \tag{2.135}$$

the so called *conservative temperature* θ^* is defined, (the motivation for this choice is the identity of potential, conservative and in-situ temperature for ocean water with $S = 35$,)

$$\theta^* = \frac{H^0}{c_p^*}. \tag{2.136}$$

and

$$c_p^* \rho \frac{D\theta^*}{Dt} = \left(\frac{\theta}{T} \left(\frac{\partial H}{\partial S} \right)_{p\eta} - \left(\frac{\partial H^0}{\partial S} \right)_{p\eta} \right) \mathbf{\nabla} \cdot \mathbf{J}_S + \frac{\theta}{T} \left(-\mathbf{\nabla} \mathbf{J}_H + \rho \epsilon \right). \tag{2.137}$$

θ^* is strictly conserved for adiabatic conditions. Hence, it can be used as a state variable like the entropy. For practical application further simplifications are possible:

- the potential temperature and the in-situ temperature differ only slightly. Since they are given in the Kelvin scale the approximation $\frac{\theta}{T} \approx 1$ is possible.
- the difference in the dependency of H and H^0 on salinity is small. Hence, the influence of the salinity flux on the conservative temperature is small.
- heating by dissipation of mechanical energy is small compared with other heat sources.

Hence, for the conservative temperature the equation

$$c_p^* \rho \frac{D\theta^*}{Dt} = -\mathbf{\nabla} \mathbf{J}_H, \tag{2.138}$$

is an excellent approximation. This is used in most modern computer codes to solve the hydrodynamic equations for the ocean numerically.

2.5.3. The potential temperature

Moving a fluid element adiabatically, the entropy remains unchanged, but the temperature may change. Hence, we can ask for the temperature a water parcel would have, if moved adiabatically to another depth level. This temperature, the *potential temperature* is a conservative property of the water parcel. This is very important in oceanography, it allows the comparison and distinction of water masses from its conservative properties, even if they are in different depth. This concepts is fundamental in oceanography since it helps to identify water masses and to follow their spreading throughout the ocean basins. Both together are like a more or less unique fingerprint of a water mass which is changed only slowly by mixing with other water masses.

Mostly, potential temperature is defined using the entropy,

$$\eta(S,T,p) = \eta(S,\theta,p^0) = \eta^0(S,\theta). \tag{2.139}$$

This is an implicit definition of θ, the conservative property in the background is the entropy. p^0 is a reference pressure, previously introduced as the sea surface pressure. This limititation is not needed, p^0 is an arbitrary pressure level the fluid parcels are moved to.

For θ an advection-diffusion equation can be derived. The derivation is long and is not given here.

$$\rho c_p^0 \frac{D\theta}{Dt} = \frac{\theta}{T}\left(\rho\epsilon - \boldsymbol{\nabla}\cdot\mathbf{J}_H\right) + \frac{\theta}{T}\left(\mu - T\frac{\partial\mu^0}{\partial T}\right) \tag{2.140}$$

μ is the chemical potential, μ^0 the chemical potential at reference pressure p^0. Again, a much simpler approximation can be derived,

$$\rho c_p^0 \frac{D\theta}{Dt} = -\boldsymbol{\nabla}\cdot\mathbf{J}_H. \tag{2.141}$$

This looks similar like that for the conservative temperature, but is less exact, since the specific heat depends on temperature, pressure and salinity. Nevertheless, for historical reasons the potential temperature is much more common in oceanography that the conservative temperature.

The density

$$\hat{\rho}(S,\theta) = \rho(S,\theta,p^0) \tag{2.142}$$

is called *potential density*.

All material properties can be also written in terms of the conservative temperature:

$$\alpha^* = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\theta^*}\right)_{Sp} \tag{2.143}$$

$$\gamma^* = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{theta^*p} \tag{2.144}$$

$$\kappa^* = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{Stheta^*} \tag{2.145}$$

2.6. Hydrostatic fluids

2.6.1. Equilibrium conditions

The only force acting at a fluid at rest is the gravitational force represented by a gravitation potential Φ. Obviously a fluid does not collapse within a gravitational field and the force per fluid element must be balanced by internal forces. Figure 2.6 shows a sketch of these forces. Considering a fluid element, there must exist a gradient in the pressure field balancing locally the external forces. The forces between the fluid elements can be calculated by means of quantum statistics starting from the interaction of the fluids constituents.

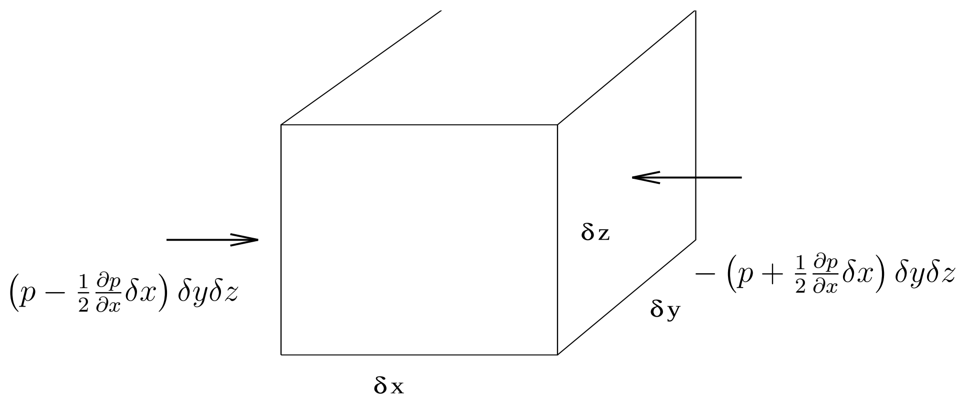


Figure 2.6. Effective pressure force on a volume element from a pressure gradient

This concept is complicated but suitable to understand the reason for the observed pressure force. For practical applications in oceanography or generally in fluid dynamics a simpler approach is helpful. Gravitation force can be written as gradient of the geopotential

$$\mathbf{g} = -\boldsymbol{\nabla}\Phi. \tag{2.146}$$

This leads us to the equilibrium condition

$$\rho\boldsymbol{\nabla}\Phi + \boldsymbol{\nabla}p = 0. \tag{2.147}$$

It will be shown later, that the geopotential includes the centrifugal force due to the earth's rotation. Hence, in the absence of other forces, the "downward" - direction is defined by the gradient of Φ. Calling this co-ordinate z, we find the hydrostatic equations,

$$\frac{\partial p}{\partial z} = -\rho g. \tag{2.148}$$

Both, ρ and p depend solely on z and not on the the horizontal co-ordinates. Hence, if an approximation for ρ is known, an expression for p can be found from vertical integration. Also the density profile can be derived from α, γ and the sound velocity. The total differential of the density reads

$$\begin{aligned} \frac{d\rho}{dz} &= \frac{\partial\rho}{\partial T}\frac{dT}{dz} + \frac{\partial\rho}{\partial S}\frac{dS}{dz} + \frac{\partial\rho}{\partial p}\frac{dp}{dz} \\ &= -\rho\alpha\frac{dT}{dz} + \rho\gamma\frac{dS}{dz} - \rho\frac{g}{c^2}. \end{aligned} \tag{2.149}$$

For oceanic conditions, it turns out that the density is determined mostly by the compressibility. Neglecting for the moment the dependency on temperature and salinity, the density equation reads

$$\frac{d\rho}{dz} \approx -\rho\frac{g}{c^2}. \tag{2.150}$$

This has the formal solution

$$\begin{aligned} \rho(z) &= \rho(0)exp\int_z^0 dz\frac{g}{c^2} \\ &\approx \rho(0)e^{-z\frac{g}{c^2}} \\ &\approx \rho(0)\left(1 - z\frac{g}{c^2}\right). \end{aligned} \tag{2.151}$$

The corresponding pressure reads

$$\begin{aligned} p(z) &= p(0) + c^2\rho(0)\left(e^{-z\frac{g}{c^2}} - 1\right) \\ &\approx p(0) - \rho(0)gz\left(1 - \frac{zg}{2c^2}\right). \end{aligned} \tag{2.152}$$

For $c^2 \to \infty$ the well known approximation $p(z) \approx \rho gh$ (h: height of the water column) is verified.

2.6.2. Stability of the water column

Hydrostatic equilibrium does not imply the stability of this equilibrium state against small fluctuations. If ocean density is increasing downward, lowering of a surface water parcel implies rising of a neighbouring water parcel from below. This way the center of gravity of the ocean is elevated upwards, the potential energy is enhanced. If the density is decreasing downward, an downward elevation of a surface water parcel diminishes the potential energy. Hence, the equilibrium state does not have a minimum of potential energy and kinetic energy may be gained from potential energy. A convective flow is likely to develop, the stratified water body is not stable.

However, this is an oversimplified consideration. Water, pushed downward undergoes compression and adiabatic heating (heat transfer is considered as small). Hence, the changes of potential energy may be modified by changing density and the distinction of stable and unstable stratification requires more thoughts.

We consider adiabatic motion of fluid parcels. Entropy of these water parcels is unchanged. The total differential of the entropy in terms of pressure and temperature reads

$$T d\eta = T \left(\frac{\partial \eta}{\partial T}\right)_{pS} dT + T \left(\frac{\partial \eta}{\partial p}\right)_{TS} dp. \tag{2.153}$$

Salinity of the water parcel is constant. The first coefficient is related to the specific heat,

$$c_p = T \left(\frac{\partial \eta}{\partial T}\right)_{pS}, \tag{2.154}$$

With the help of the Maxwell relation

$$\left(\frac{\partial \eta}{\partial p}\right)_{TS} = -\left(\frac{\partial v}{\partial T}\right)_{pS}, \tag{2.155}$$

the second coefficient can be expressed in terms of the thermal expansion coefficient,

$$\begin{aligned} T \left(\frac{\partial v}{\partial T}\right)_{pS} &= -\frac{T}{\rho^2} \left(\frac{\partial \rho}{\partial T}\right)_{pS} \\ &= \frac{T\alpha}{\rho}. \end{aligned} \tag{2.156}$$

Hence, each change in pressure corresponds to a temperature change,

$$dT = \frac{\alpha T}{\rho c_p} dp. \tag{2.157}$$

The quantity

$$\Gamma = \frac{\alpha T}{\rho c_p}, \tag{2.158}$$

is called *adiabatic lapse rate*. It can be shown, that

$$\Gamma = \frac{\alpha T}{\rho c_p} = \frac{\alpha^* \theta^*}{\rho c_p^*}, \tag{2.159}$$

This provides a formula to calculate the potential temperature

$$\theta^*(p_0) = T(p) + \int_p^{p_0} \Gamma(S, \theta^*(p_0), p) dp. \tag{2.160}$$

Now we consider the density change from the compression or expansion, when the water parcel is moves down- or upwards,

$$\delta \rho = \left(\frac{\partial \rho}{\partial T}\right)_{pS} dT + \left(\frac{\partial \rho}{\partial p}\right)_{TS} dp. \tag{2.161}$$

For an adiabatic elevation, the temperature change is related to the pressure change by Eq. 2.157,

$$\delta\rho = \left(\left(\frac{\partial\rho}{\partial p}\right)_{TS} - \frac{\alpha^2 T}{c_p}\right)\frac{dp}{dz}dz. \tag{2.162}$$

The adiabatic and isohaline pressure change is related to the sound velocity,

$$\left(\frac{\partial\rho}{\partial p}\right)_{\eta S} = \frac{1}{c^2}, \tag{2.163}$$

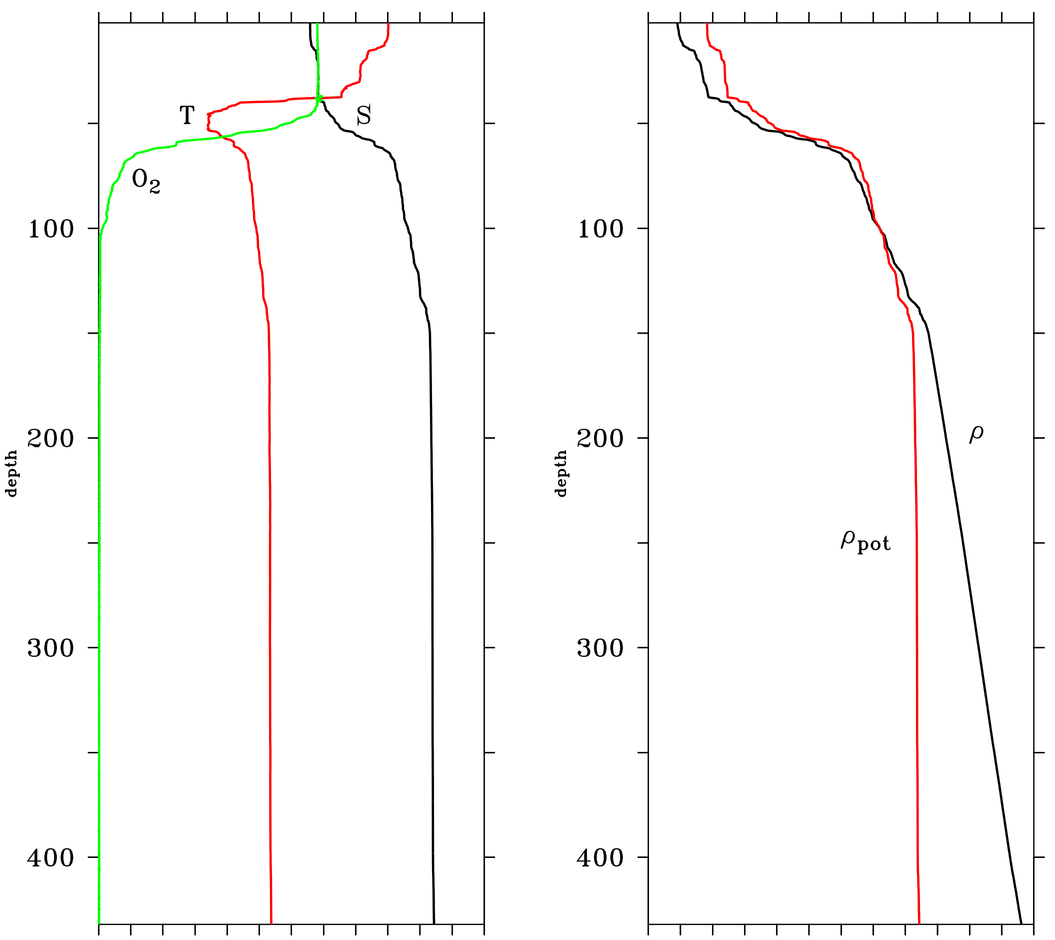


Figure 2.7. Temperature, salinity and density in the Landsort Deep, Nov. 2013

So, with the help of the hydrostatic equation,

$$\frac{\partial p}{\partial z} = -\rho g, \tag{2.164}$$

we get in turn

$$\begin{aligned}\delta\rho &= -g\rho\left(\left(\frac{\partial\rho}{\partial p}\right)_{TS} - \frac{\alpha^2 T}{c_p}\right)dz \\ &= -\frac{g\rho}{c^2}dz.\end{aligned} \tag{2.165}$$

The density change just due to the vertical ocean stratification is

$$d\rho = \frac{d\rho}{dz}dz = \left(\left(\frac{\partial\rho}{\partial T}\right)_{pS}\frac{dT}{dz} + \left(\frac{\partial\rho}{\partial p}\right)_{TS}\frac{dp}{dz} + \left(\frac{\partial\rho}{\partial S}\right)_{Tp}\frac{dS}{dz}\right)dz. \tag{2.166}$$

For the difference of the density change from the elevation and the static density difference we find two equivalent expressions,

$$\begin{aligned}\delta\rho - d\rho &= \left(\rho g\frac{\alpha^2 T}{c_p}\frac{dp}{dz} - \left(-\rho\alpha\frac{dT}{dz} + \rho\gamma\frac{dS}{dz}\right)\right)dz \\ &= \left(\rho g\frac{\alpha^2 T}{c_p}\frac{dp}{dz} + \rho\alpha\frac{dT}{dz} - \rho\gamma\frac{dS}{dz}\right)dz,\end{aligned} \tag{2.167}$$

or

$$\delta\rho - d\rho = \left(-\frac{d\rho}{dz} - \frac{g\rho}{c^2}\right)dz. \tag{2.168}$$

The quantity N^2 with

$$N^2 = \frac{g}{\rho}\frac{\delta\rho - d\rho}{dz} = g\left(g\frac{\alpha^2 T}{c_p}\frac{dp}{dz} + \alpha\frac{dT}{dz} - \gamma\frac{dS}{dz}\right), \tag{2.169}$$

has the dimension of a frequency and is called Brunt-Väisälä frequency or buoyancy frequency. We find two similar expressions,

$$N^2 = \left(-\frac{g}{\rho}\frac{d\rho}{dz} - \frac{g^2}{c^2}\right), \tag{2.170}$$

and

$$N^2 = g\left(g\frac{\alpha^2 T}{c_p}\frac{dp}{dz} + \alpha\frac{dT}{dz} - \gamma\frac{dS}{dz}\right). \tag{2.171}$$

We will proof later, that the water column is stable for $N^2 > 0$ but becomes unstable against convection for $N^2 < 0$.

Now we can simplfy these results considerably introducing the potential temperature and the potential density. Recall, the potential density $\hat{\rho}$, is that density a water parcel would have, when elevated adiabatically. The compression effect is implicit. So the buoyancy frequency reads in terms of the potential density

$$N^2 = \left(-\frac{g}{\rho}\frac{d\hat{\rho}}{dz}\right). \tag{2.172}$$

Chapter 3

Wind driven flow

The oceans are obviously not at rest. Surface currents, partially strong and dangerous for ships are known since centuries. They are known to be driven mostly by the inhomogeneity of the gravitation field of moon and sun. But also winds, namely the trade winds may drive currents by the shear force exerted by the wind blowing over the sea. Due to the different density of water and air, water density is about 1000kg m^-3, air density only about 1.275kg m^-3, the speed of wind driven surface flow is much smaller than the typical air speed.

The interaction between air and water is complicated, namely the mechanisms of transfer of momentum, energy and matter through the air-sea interface are still subject of research.

If wind blows over the ocean, the most obvious result is a growing wave field. It is a superposition of wave tracks with varying frequency, wave number and wave height. These waves are carrying energy and momentum. If waves are breaking, energy is lost and transformed into heat. Momentum is not lost but stays in the ocean a process that gives rise to a surface current distributed over a much larger are than the single breaking surface wave.

It is this surface current we are interested in here. Roughly spoken, it is a wind driven current, in more detail it is the result of a complex interaction of wind, waves and currents. For the moment we neglect these details and ask for the relation between the wind and the large scale flow. This requires a Reynold-averaging of the equations of motion. Turbulence is hidden in effective quantities for ocean viscosity and mixing and for the air-sea interaction.

3.1. Early theory - the Zöppritz ocean

We reconsider an early theory of wind driven ocean currents. Based on the hydrodynamic equations, but without inclusion of turbulence, vertical velocity profiles develop due to the vertical momentum flux from friction. The theory is often called “failed”. However, it is still of some interest, since it clearly demonstrates the necessity to include turbulent friction into considera-

tion.

Some years ago, say around 1870, it was questioned by oceanographers if winds may drive deep ocean currents. The typical ocean depth as well as currents in the deep ocean were widely unknown and a theory of ocean currents was far from being developed. Nevertheless, the basic equations of fluid dynamics were discovered and were waiting to be solved to investigate ocean dynamics. The general outlines of tide theory based on Newtons and Laplace's ideas are known. On the other hand the mathematical tools to solve partial differential equations were developed and have been applied successfully to mechanical and optical problems. 1878 Karl Jakob Zöppritz (Gießen) published a first solution of the hydrodynamic equations for wind driven currents. He starts from simplified hydrodynamic equations, (incompressible)

$$\begin{aligned} \frac{\partial \rho \mathbf{u}}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}\mathbf{u}) &= -\boldsymbol{\nabla} p + \eta \boldsymbol{\nabla}^2 \mathbf{u} + \rho \mathbf{f}, \\ \boldsymbol{\nabla} \cdot \mathbf{u} &= 0. \end{aligned} \tag{3.1}$$

Density is considered as a constant, the only external force is gravity that is directed downward,

$$\mathbf{f} = (0, 0, -g). \tag{3.2}$$

The Reynolds averaging was unknown η is the molecular viscosity.

To solve these equation, surface- and bottom boundary conditions are needed. These are discussed in terms of molecular friction between two fluids, the cross surface velocity between air and water must be continuously, (air and water are always in touch)

$$\mathbf{u}^{air} - \mathbf{u}^{water} = 0, \qquad z = 0. \tag{3.3}$$

At the bottom the normal and the tangent velocity vanish, the tangent first because the flow into the bottom is zero, the second because water is fully in contact with the resting sea floor.

Assuming an initially flat sea surface, an uniform wind and a flat bottom, there is no mechanism to disturb this symmetry. This implies also that the vertical velocity vanishes. Hence, the third momentum equation defines the pressure,

$$\begin{aligned} \frac{\partial p}{\partial z} &= -\rho g, \\ p(z) &= p^{air} - \int_0^z dz' \rho g = p^{air} - \rho g z. \end{aligned} \tag{3.4}$$

Note, $z < 0$! This is independent off the horizontal coordinates and the horizontal pressure gradients vanish.

Exercise 3.10 Show, that in this case also the non-linear terms in the momentum equation disappear!

We assume the wind blows in x-direction. The velocity in y direction remains zero and needs not to be considered. So the final equation is simple,

$$\rho \frac{\partial u}{\partial t} = \eta \frac{\partial^2 u}{\partial z^2}. \tag{3.5}$$

Finally, we need the boundary condition for the momentum flux between air and water. Within the water the vertical flux of horizontal momentum is

$$\tau^{water} = \eta \frac{\partial u}{\partial z}, \qquad z = 0. \tag{3.6}$$

In the air it has the value τ^{air} and is some function of the wind speed. The momentum flux must be a steady function, otherwise momentum conservation would be violated,

$$\tau^{air} = \eta \frac{\partial u}{\partial z}, \qquad z = 0. \tag{3.7}$$

Hence, we deal with the simple problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nu \frac{\partial^2 u}{\partial z^2}, \\ u(0, z) &= 0 \quad \text{for} \quad t < 0, \\ u(t, z) &= 0 \quad \text{for} \quad z = -H, \\ \nu \frac{\partial u}{\partial z} &= \tau(t) \quad \text{for} \quad z = 0. \end{aligned} \tag{3.5}$$

$\nu = \eta/\rho$ is the kinematic viscosity, $\tau = \tau^{air}/\rho$.

As the first step we ask for a steady state solution eventually established after a long time for a constant wind stress. The corresponding equation is

$$\begin{aligned} \nu \frac{\partial^2 u}{\partial z^2} &= 0, \\ u(t, z) &= 0 \quad \text{for} \quad z = -H, \\ \nu \frac{\partial u}{\partial z} &= \tau \quad \text{for} \quad z = 0. \end{aligned} \tag{3.4}$$

The first integral reads

$$\nu \frac{\partial u}{\partial z} = c_1, \tag{3.5}$$

where the surface boundary condition defines the constant,

$$c_1 = \frac{\tau}{\nu}. \tag{3.6}$$

The second integral gives

$$u(z) = \frac{\tau}{\nu} z + c_2, \tag{3.7}$$

where the bottom boundary condition defines c_2. The solution reads

$$u(z) = \frac{\tau}{\nu}(z+H) = u(0)\frac{z+H}{H}, \tag{3.8}$$

the vertical profile is linear and independent off the viscosity. The vertical momentum flux νu_z is constant through the water column, the momentum coming into the ocean at the sea surface leaves the ocean through the sea floor. This is our first example for an ocean momentum budget. There exists a steady solution for a constant force! Internal friction in connection with bottom friction establishes the balance. Indeed, if the ocean is very deep $H \to \infty$, $u(0)$ diverges. The balance is never established, the ocean can take up infinite momentum since there is no momentum sink any more.

Now we consider the time dependent solution. Zöppritz uses another surface boundary condition and solves the time dependent problem by a Fourier series decomposition. We use another approach based on a Green's function defined by

$$-\frac{\partial G(t-t',z,z')}{\partial t'} - \nu\frac{\partial^2 G(t-t',z,z')}{\partial z'^2} = \delta(t-t')\delta(z-z'). \tag{3.9}$$

Boundary and initial conditions are not defined - a proper choice will help to simplify the solution of our problem. Writing bot equations in a shorter notation,

$$\begin{aligned} u_{t'} - \nu u_{z'z'} &= 0, \\ -G_{t'} - \nu G_{z'z'} &= \delta(t-t')\delta(z-z'), \end{aligned} \tag{3.9}$$

the first equation can be multiplied with G and the second with u and the first can be subtracted from the second one,

$$-uG_{t'} - Gu_{t'} - \nu\left(uG_{z'z'} - Gu_{z'z'}\right) = u\delta(t-t')\delta(z-z'), \tag{3.10}$$

or slightly rewritten,

$$u\delta(t-t')\delta(z-z') = -(uG)_{t'} - \nu\left(uG_{z'} - Gu_{z'}\right)_{z'}. \tag{3.11}$$

This equation can be integrated using the property of the δ-distributions,

$$\int_{\infty}^{\infty} dx \delta(x-x') f(x') = f(x), \tag{3.12}$$

$$\begin{aligned} u(t,z) &= \int_{\infty}^{\infty} dt' \int_{-H}^{0} dz' \left(-(uG)_{t'} - \nu\left(uG_{z'} - Gu_{z'}\right)_{z'}\right) \\ &= -\int_{-H}^{0} dz'\ u(t')G(t-t')|_{t'=-\infty}^{t'=\infty} - \nu \int_{\infty}^{\infty} dt'\ \left(uG_{z'} - Gu_{z'}\right)|_{z'=-H}^{z'=0}. \end{aligned} \tag{3.12}$$

Let us consider the first term. For $t' > t$ somehow the flow for times larger than t would influence the result. This violates causality. Hence, we must require

$$G(t-t') = 0 \quad \text{for} \quad t' > t. \tag{3.13}$$

Together with the initial condition (ocean at rest for $t < 0$) the first term disappears. In the second term u is known at the bottom, the derivative at the surface. Hence, specifying the boundary conditions,

$$\begin{aligned} G'_z &= 0 \quad \text{for} \quad z = 0, \\ G &= 0 \quad \text{for} \quad z = -H, \end{aligned} \tag{3.13}$$

all unknown are eliminated

$$u(t,z) = \int_{\infty}^{\infty} dt' G(t-t', z, 0)\tau(t'). \tag{3.14}$$

Knowing the Green's function the velocity for an arbitrary wind stress is known from the convolution integral Eq. (3.14).

The Fourier transformed of the Green's function is

$$G(t-t', z) = \int_{-\infty}^{-\infty} \frac{d\omega}{2\pi} \sum_m e^{-i\omega(t-t')} e^{imz} G_m(\omega) \tag{3.15}$$

The sum run over all positive and negative m. From the surface boundary condition the relation

$$G_m = -G_{-m}, \tag{3.16}$$

is found. The bottom boundary condition can only be satisfied by

$$G_0 + \sum_{m=1}^{\infty} G_m \left(e^{imH} + e^{-imH} \right) = 0 \tag{3.17}$$

The G_m are arbitrary functions of ω. Hence, the bottom boundary condition must be valid for all summands separately,

$$\begin{aligned} G_0 &= 0 \\ \left(e^{imH} + e^{-imH} \right) &= 2\cos(mH) = 0. \end{aligned} \tag{3.17}$$

Hence, the functions m must be

$$mH = \left(n + \frac{1}{2} \right) \pi = \frac{1}{2}(2n+1)\pi. \tag{3.18}$$

The following ansatz for the Green's function is compatible with the boundary conditions

$$G(t-t', z) = \int_{-\infty}^{-\infty} \frac{d\omega}{2\pi} \sum_{n=0}^{\infty} e^{-i\omega(t-t')} G_n(\omega) F_n(z) \tag{3.19}$$

The functions

$$F_n(z) = \sqrt{\frac{2}{H}} \cos\left((2n+1)\pi \frac{z}{2H} \right) \tag{3.20}$$

are normalized,

$$\int_{-H}^{0} dz F_n(z) F_l(z) = \delta_{n,l}. \tag{3.21}$$

Now we insert this expression in the equation for G and find

$$\int_{-\infty}^{-\infty} \frac{d\omega}{2\pi} \sum_{n=0}^{\infty} \left(-i\omega + \nu \left(\frac{(2n+1)\pi}{2H} \right)^2 \right) e^{-i\omega(t-t')} G_n(\omega) F_n(z) = \delta(t-t')\delta(z) \tag{3.22}$$

Now we multiply with $e^{i\bar{\omega}t}$ and F_l and integrate over z and t. We use the normalization of the functions F_n and the relation

$$\int_{-\infty}^{\infty} dt e^{i\omega t} = 2\pi\delta(\omega), \tag{3.23}$$

$$\left(-i\omega + \nu \left(\frac{(2n+1)\pi}{2H} \right)^2 \right) G_n(\omega) = \sqrt{\frac{2}{H}}. \tag{3.24}$$

This gives us the Green's function in the Fourier space:

$$G_n(\omega) = \sqrt{\frac{2}{H}} \frac{i}{\left(\omega + i\nu \left(\frac{(2n+1)\pi}{2H} \right)^2 \right)}. \tag{3.25}$$

The Greens function has poles in the lower complex half plane. This can be used to perform the inverse Fourier transformation,

$$G_n(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G_n(\omega) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sqrt{\frac{2}{H}} \frac{i e^{-i\omega(t-t')}}{\left(\omega + i\nu \left(\frac{(2n+1)\pi}{2H} \right)^2 \right)}. \tag{3.26}$$

The integral can be written as a contour integral in the complex plane by adding a half circle in the upper or lower half plane. The exponent is for an imaginary $\omega = i\tilde{\omega}$

$$-i\omega(t-t') = \tilde{\omega}(t-t'). \tag{3.27}$$

For a positive exponent the exponential is large, otherwise it becomes infinitely small, when the integration path is chosen to be far from the origin. Only in the latter case the contour integral can be closed. Hence, for $t > t'$ the integral can be closed in the lower half plane and encloses the poles at

$$\omega = -i\nu \left(\frac{(2n+1)\pi}{2H} \right)^2. \tag{3.28}$$

For $t < t'$ the integral path can be closed in the upper half plane, where, however, singularities do not exist. From Cauchy's integral theorem

$$f(z) = \frac{1}{2\pi i}\oint d\xi \, \frac{f(\xi)}{\xi - z}, \qquad \text{anti - clockwise} \tag{3.29}$$

we find

$$\begin{aligned} G_n(t-t') &= \sqrt{\frac{2}{H}}\theta(t-t')e^{-\nu k_n^2(t-t')} \\ .k_n &= \frac{(2n+1)\pi}{2H} \end{aligned} \tag{3.29}$$

The full Green's function is

$$G(t-t',z) = \sum_{n=0}^{\infty} \frac{2}{H}\theta(t-t')e^{-\nu k_n^2(t-t')}\cos(k_n z), \tag{3.30}$$

and the velocity is

$$\begin{aligned} u(t,z) &= \int_{\infty}^{\infty} dt' G(t-t',z,0)\tau(t') \\ &= \int_0^t dt' \sum_{n=0}^{\infty} \frac{2}{H} e^{-\nu k_n^2(t-t')}\cos(k_n z)\,\tau(t'). \end{aligned} \tag{3.30}$$

As an example a constant wind stress τ. In this case the time integral can be carried out,

$$u(t,z) = \sum_{n=0}^{\infty} \frac{2\tau}{H\nu k_n^2} e^{-\nu k_n^2 t}\cos(k_n z)\left(e^{\nu k_n^2 t} - 1\right) \tag{3.31}$$

For large time this converges to

$$u(t\to\infty,z) \to \sum_{n=0}^{\infty} \frac{2\tau}{H\nu k_n^2}\cos(k_n z), \tag{3.32}$$

which is essentially a sum of the type

$$\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2} = \frac{\pi}{8}(\pi - 2|x|). \tag{3.33}$$

So the sum can be carried out

$$u(t\to\infty,z) \to \frac{H\tau}{\nu}\left(1 - \frac{|z|}{H}\right) = \frac{\tau}{\nu}(H+z), \tag{3.34}$$

exactly the linear profile as derived previously. The time dependent solution is

$$u(t,z) = \frac{\tau}{\nu}(H+z) - \sum_{n=0}^{\infty} \frac{2\tau}{H\nu k_n^2} e^{-\nu k_n^2 t} \cos(k_n z). \tag{3.35}$$

Unfortunately, this sum leads to a Jacobi elliptic function and the further analytical treatment is difficult. The sum converges rapidly for large time but here approximately the asymptotic solution is known. For $t = 0$ the sum cancels the first term, but the next order in t the calculation becomes difficult.

To find an alternative way for to evaluate the velocity, we consider the difference between the k_n:

$$\Delta k = k_{n+1} - k_n = \frac{\pi}{2H}(2(n+1) - 2n) = \frac{\pi}{H}. \tag{3.36}$$

We can rewrite the Green's function

$$\begin{aligned} G(t-t',z) &= \sum_{n=0}^{\infty} \frac{2}{H}\theta(t-t') e^{-\nu k_n^2 (t-t')} \cos(k_n z) \\ &= \int_0^{\infty} dk \frac{2}{\pi}\theta(t-t') e^{-\nu k^2 (t-t')} \cos(kz) \end{aligned} \tag{3.36}$$

From integral tables we find for the integral

$$\int_0^{\infty} dx e^{-ax^2} \cos(bx) = \sqrt{\frac{\pi}{4a}} e^{-\frac{b^2}{4a}}, \tag{3.37}$$

and the Green's function for large H reads

$$G(t-t',z) = \theta(t-t') \frac{1}{\sqrt{\pi\nu(t-t')}} e^{-\frac{z^2}{4\nu(t-t')}} \tag{3.38}$$

The velocity is (substituting $s = t - t'$)

$$u(t,z) = = \frac{\tau}{\sqrt{\pi\nu}} \int_0^t ds \frac{1}{\sqrt{s}} e^{-\frac{z^2}{4\nu s}}, \tag{3.39}$$

To simplify the remaining integral we substitute

$$q = \frac{1}{\sqrt{s}}, \tag{3.40}$$

and find with

$$\frac{ds}{\sqrt{s}} = -\frac{2}{q^2} dq = 2\frac{d}{dq}\left(\frac{1}{q}\right) dq \tag{3.41}$$

The second identity helps to integrate by parts.

$$
\begin{aligned}
u(t,z) &= \frac{\tau}{\sqrt{\pi\nu}} \int_{\infty}^{\frac{1}{\sqrt{t}}} dq 2 \frac{d}{dq}\left(\frac{1}{q}\right) e^{-\frac{z^2q^2}{4\nu}} \\
&= \frac{2\tau}{\sqrt{\pi\nu}} \frac{1}{q} e^{-\frac{z^2q^2}{4\nu}} |_{\infty}^{\frac{1}{\sqrt{t}}} - \frac{2\tau}{\sqrt{\pi\nu}} \int_{\infty}^{\frac{1}{\sqrt{t}}} dq \frac{(-2z^2q)}{4\nu q} e^{-\frac{z^2q^2}{4\nu}} \\
&= \frac{2\tau}{\sqrt{\pi\nu}} \sqrt{t} e^{-\frac{z^2}{4\nu t}} + \frac{2\tau}{\sqrt{\pi\nu}} \int_{\infty}^{\frac{1}{\sqrt{t}}} dq 2 \frac{(z^2)}{4\nu} e^{-\frac{z^2q^2}{4\nu}}
\end{aligned} \tag{3.39}
$$

This integral can be traced back to a well defined function, the error function

$$
\begin{aligned}
\operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x dx' e^{-x'^2}, \\
1 - \operatorname{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_x^{\infty} dx' e^{-x'^2}.
\end{aligned} \tag{3.39}
$$

Substituting again

$$
\frac{z^2q^2}{4\nu} = x^2 \tag{3.40}
$$

we find

$$
u(t,z) = \frac{2\tau}{\sqrt{\pi\nu}} \sqrt{t} e^{-\frac{z^2}{4\nu t}} - \frac{\tau z}{\nu} \left(1 - \operatorname{erf}\left(\sqrt{\frac{z^2}{4\nu t}}\right)\right) \tag{3.41}
$$

3.2. The Ekman theory

The Ekman theory is the most simple valid appoach to understand wind driven currents in a rotating ocean. It relates the vertical (downward) flux of horizontal momentum to a turbulent viscosity. The theory allows for understanding the surface Ekman transport but also of inertial oscillations within the surface layer.

3.2.1. The classical Ekman solution

The classical Ekman[1] theory is not part of this lecture but is given in most textbooks on oceanography. One of the basic ideas of the Ekman theory compared with the Zöppritz approach based on molecular friction is the application of a turbulent viscosity. It allows

[1]Vagn Walfrid Ekman, * 3. MaY 1874 in Stockholm; †9. March 1954 in Gostad. Oslo, Lund, Göteborg

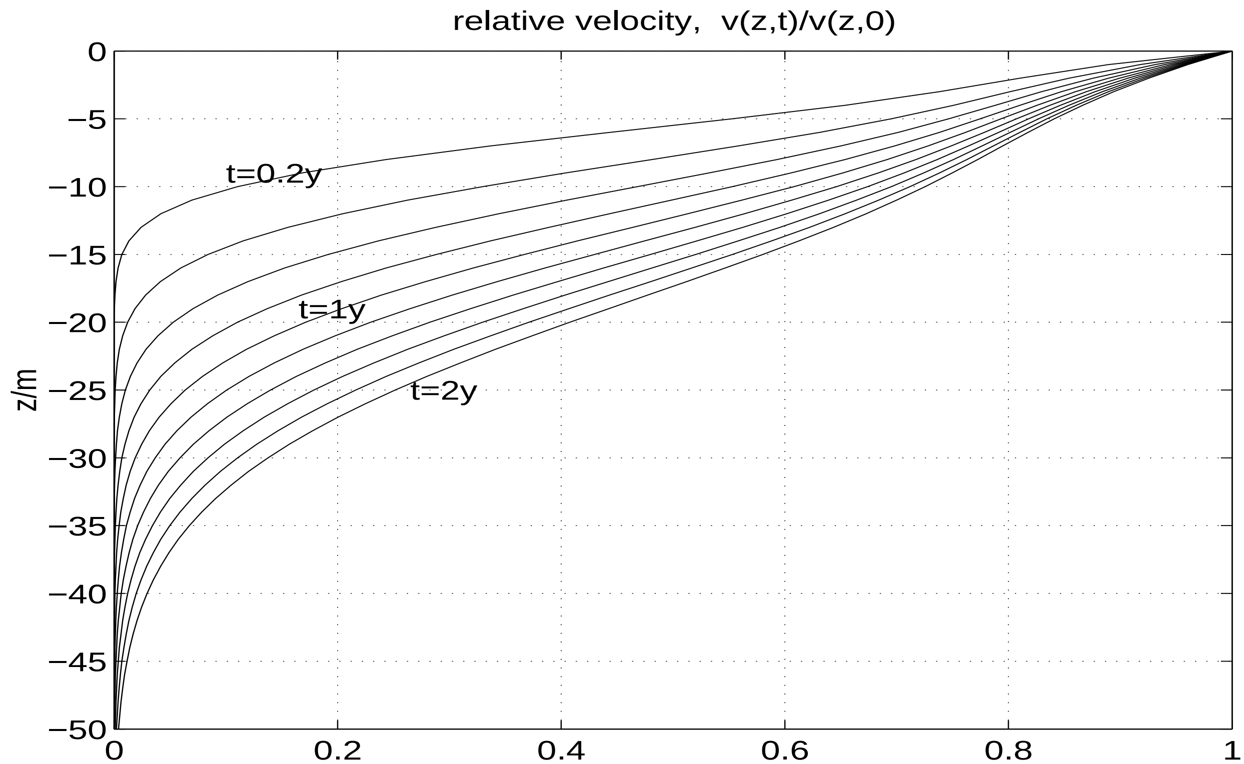


Figure 3.1. Relative velocity for a constant wind over an ocean with molecular friction only.

to express the turbulent vertical momentum flux in terms of the "large scale" velocities. The vertical flow of large scale momentum is considered as a "down-gradient" flux,

$$\overline{u'w'} \approx -A_v \frac{\partial u}{\partial z}. \tag{3.42}$$

There are two fundamental results of this theory,

- the existence of an average mass transport almost perpendicularly to the surface wind stress overlaid by inertial oscillations. This finding can be considered as well verified by field measurements.

- the depth dependence of mass transport is dominated by the so called "Ekman spiral" which is, however, found only in some rare field data after special time averaging.

3.2.2. The concept of volume forces

Alternatives for the Ekman theory start from the assumption that the momentum flux between atmosphere and ocean is primarily related to small scale processes like surface gravity waves, which in turn generates more turbulence by non-linear wave-wave interaction and wave breaking, but also Langmuir circulation and heat-flux driven convection may play an important role. Surface waves carry also a large scale momentum. This momentum stays in the ocean, if the wave energy is dissipated by internal friction or transfer of wave energy into potential energy by lifting up the center of gravity of water columns due to mixing.

Exercise 3.11 Consider two fluid layers with different velocity u_1 and u_2. Assume strong turbulent mixing between both layers that tends to equilibrate the velocity difference between both the layers. Show this considering the momentum equations, make appropriate assumptions on linearity, pressure term and so on. Calculate to total kinetic energy density as well as the total momentum of the two layer system. Make a statement on momentum and energy conservation. Remark: This exercise is a demonstration of a principle. So the assumption that the turbulent momentum flux with other layers is negligible is justified.

The result is the known divergence of the vertical (mostly downward) flux of horizontal momentum,

$$\begin{aligned}\frac{\partial \overline{u}}{\partial t} - f\overline{v} + p_x &= -\frac{\partial}{\partial z}\overline{u'w'},\\ \frac{\partial \overline{v}}{\partial t} + f\overline{u} + p_y &= -\frac{\partial}{\partial z}\overline{v'w'}.\end{aligned} \tag{3.41}$$

Small scale processes bring the momentum downward. In each layer the large scale momentum gets a surplus from the small scale processes, when energy of those processes is dissipated.

Considering the large scale processes, this momentum transfer from small to large scale processes by dissipation of small scale energy acts like a momentum source. From measurements and theory it is known, that small scale surface processes do not penetrate arbitrarily deep but are confined to a mixing (to be distinguished carefully from the "mixed" layer) surface layer of thickness H_{mix}. H_{mix} is not defined by the large scale processes itself but is treated as a parameter of the theory of large scale currents. Also the vertical shape is not defined by the large scale flow but is a result of the small scale processes.

As the bottomline of this discussion, small scale, wind driven surface processes go along with a divergent vertical flux of horizontal momentum, which is represented in the equations of the large scale momentum like a source (X, Y) confined to a mixing layer depth H_{mix},

$$\begin{aligned}\frac{\partial \overline{u}}{\partial t} - f\overline{v} + p_x &= X,\\ \frac{\partial \overline{v}}{\partial t} + f\overline{u} + p_y &= Y.\end{aligned} \tag{3.41}$$

The surface boundary condition for the vertical flux of horizontal momentum is unchanged,

$$-\overline{u'w'} = \frac{\tau^x}{\rho_0}, \qquad -\overline{v'w'} = \frac{\tau^x}{\rho_0}, \qquad \text{for} \quad z = 0. \tag{3.42}$$

Assuming for a moment zero momentum flux between the surface layer and the ocean below, the vertical integral of the momentum sources (volume forces) must be equal to the surface momentum flux,

$$\int_{-H_{mix}}^{0} dz\,(X, Y) = \frac{\tau^{(x,y)}}{\rho_0}. \tag{3.43}$$

Hence, the physical meaning of the volume force is the redistribution of the surface momentum flux over the surface boundary layer, namely by small scale processes like breaking surface waves, Langmuir circulation or convection.

A considerable simplification is achieved with the assumption, that the divergence of the momentum flux is constant.

Exercise 3.12 Show, that a volume flux constant over the mixed layer is equivalent to a linear profile of the vertical momentum flux. Make a plot of both quantities. Verify that the surface boundary condition is fulfilled.

In this case the volume force is constant in the mixed layer,

$$(X, Y) = \theta\,(z + H_{mix})\,\frac{\tau^{(x,y)}}{\rho_0 H_{mix}}. \tag{3.44}$$

What we should know now?

The action of the wind on the ocean surface layer can be represented by a volume force acting in this layer. The force is equivalent to the divergency of a vertical (downward) momentum flux.

3.2.3. Ekman theory with volume forces

The formal solution

We reconsider the example of an ocean driven by a uniform wind, but now by using the volume force approximation. If the wind is uniform, pressure contributions as well as non-linear terms (for the large scale only!) disappear. The vertical momentum flux below the mixing layer is small and we consider the almost inviscid case here. To investigate the influence of friction we use the Rayleigh approximation: each motion goes along with a damping proportionally to the velocity.

Exercise 3.13 Find the solution for the Rayleigh damped velocity with

$$\frac{\partial u}{\partial t} + ru = 0, \tag{3.45}$$

where an initial value $u(t = 0) = u_0$ is assumed!

The equations of motion to be solved are now (dropping the overbars and using the shorter notation u_t for the time derivative),

$$\begin{aligned} \frac{\partial u}{\partial t} + ru - fv &= X, \\ \frac{\partial v}{\partial t} + rv + fu &= Y. \end{aligned} \tag{3.45}$$

r describes the Rayleigh damping. As initial condition we assume an ocean of rest, $(u,v)(t=0) = (u_0, v_0) = 0$. Both equations can be combinded into single equation,

$$u_{tt} + 2ru_t + \left(r^2 + f^2\right) u = X_t + rX + fY. \tag{3.46}$$

This is a second order differential equation, a second initial condition is needed. Since we have eliminated v, we use the boundary condition for v in the origional equation,

$$\frac{\partial u}{\partial t} + ru = X, \quad \text{for} \quad t = 0, \tag{3.47}$$

which is the required second initial condition, now for u_t. Finally, we consider a wind "switched on" at $t = 0$,

$$X(t) = \theta(t)\tilde{X}(t) \tag{3.48}$$

Now we search for a formal solution valid for every wind forcing. To this end a Green's function $G(t, t')$ is defined,

$$G_{t't'} - 2rG_{t'} + (r^2 + f^2)G = \delta(t - t'). \tag{3.49}$$

It does not matter, to with time our experiment starts - the Green's function depends only on time differences,

$$G(t, t') = G(t - t'). \tag{3.50}$$

As the first boundary condition we require

$$G(t, t') = 0, \quad \text{for} \quad t' > t. \tag{3.51}$$

We will see below, that this corresponds to causality, no response before the reason. Multiplying the equation for u with G and that for G with u, subtracting the first from the second one and integrating over t', we find

$$\int_0^t dt'\, G_{t't'}u - Gu_{t't'} - 2r\left(G_{t'}u + uG_{t'}\right) = u(t) - \int_0^t dt'\, G(t-t')\left(X_{t'} + rX + fY\right) \tag{3.52}$$

The upper limit t comes from causality, the lower boundary from the assumption, that the ocean should be at rest for $t < 0$. Integrating by parts results in

$$\begin{aligned} u(t) &= \int_0^t dt'\, G(t-t')\left(X_{t'} + rX + fY\right) + G_{t'}u|_0^t - Gu_{t'}|_0^t - 2rGu|_0^t\,, \\ &= -\int_0^t dt'\left(G_{t'}(t-t')X(t') - G(t-t')\left(rX(t') + fY(t')\right)\right) \\ &\quad + GX|_0^t + G_{t'}u|_0^t - Gu_{t'}|_0^t - 2rGu|_0^t\,. \end{aligned} \tag{3.51}$$

The terms for $t = 0$ in the last line vanish because of the initial conditions for u and v. A last condition for the Green's function is remaining to be specified. Requiring also

$$G_{t'}(t,t') = 0, \quad \text{for} \quad t' > t. \tag{3.52}$$

all terms for $t' = t$ vanish and the formal solution reads,

$$u(t) = -\int_0^t dt'\ \left(G_{t'}(t-t') - rG(t-t')\right)X(t') - G(t-t')fY. \tag{3.53}$$

Exercise 3.14 Generalise this equation to the case of finite initial velocity.

Calculation of the Green's function

The Green's function can be calculated from a Fourier decomposition

$$G(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\omega) e^{i\omega(t'-t)}. \tag{3.54}$$

Inserting in the equation for G gives,

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} G(\omega) e^{i\omega(t'-t)}\left(-\omega^2 - 2ir\omega + f^2 + r^2\right) = \delta(t-t'). \tag{3.55}$$

Using the properties of the δ-distribution and the equivalence

$$\int_{-\infty}^{\infty} dt e^{i(\omega-\bar{\omega})t} = 2\pi\delta(\omega-\bar{\omega}), \tag{3.56}$$

$G(\omega)$ is found

$$G(\omega) = \frac{-1}{\omega^2 + 2ir\omega - f^2}. \tag{3.57}$$

The denominator of $G(\omega)$ has two zeros,

$$\omega_{1,2} = -ir \pm f, \tag{3.58}$$

and the Green's function can be rewritten like,

$$G(\omega) = \frac{-1}{2f}\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega-\omega_2}\right). \quad (3.59)$$

The Fourier integral in the Green's function can be carried out now. The best way to do this, is the transformation into a complex path integral in the complex plane closed in the lower half plane. This is a Cauchy integral

$$\begin{aligned} G(t-t') &= \oint \frac{d\omega}{2\pi i}\frac{ie^{i\omega(t'-t)}}{\omega_1-\omega_2}\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega-\omega_2}\right), \\ &= \frac{i\theta(t-t')}{2f}e^{-r(t-t')}\left(e^{-if(t-t')} - e^{if(t-t')}\right), \\ &= \frac{\theta(t-t')}{f}e^{-r(t-t')}\sin f(t-t'). \end{aligned} \quad (3.58)$$

Exercise 3.15 Execute the Fourier integral in detail by yourself!

Exercise 3.16 Verify the boundary condition for the Green's function for $t = t'$!

Constant wind forcing

For constant wind forcing, the velocity u can be calculated directly,

$$\begin{aligned} u(t) &= -\int_0^t dt'\,\left(G_{t'}(t-t') - rG(t-t')\right)X(t') - G(t-t')fY, \\ &= \int_0^t dt'\,e^{-r(t-t')}\left(\cos f(t-t')X + \sin f(t-t')Y\right). \end{aligned} \quad (3.58)$$

To understand the basics of the solution investigation of the special case $Y = 0$ is sufficient. With the integral

$$\begin{aligned} \int_0^t dse^{-rs}\cos fs &= \frac{e^{-rs}}{f^2+r^2}\left(f\sin fs - r\cos fs\right)\big|_0^t \\ &= \frac{1}{f^2+r^2}\left(e^{-rt}\left(f\sin ft - r\cos ft\right) + r\right) \end{aligned} \quad (3.58)$$

we find

$$\begin{aligned} u(t) &= \frac{X}{f^2+r^2}\left(e^{-rt}\left(f\sin ft - r\cos ft\right) + r\right), \\ v(t) &= \frac{X}{f^2+r^2}\left(-f + e^{-rt}\left(f\cos ft + r\sin ft\right)\right). \end{aligned} \quad (3.58)$$

For small friction $r \approx 0$ the result is similar to the original Ekman theory. There is a steady flow to the right (left at the southern hemisphere), the transport is

$$M_E^y = -\frac{\tau^x}{\rho f}. \tag{3.59}$$

But with increasing friction there arises a component in the direction of the wind stress.
Exercise 3.17 Verify this result!

The oscillating part is also similar, but friction slowly damps it. After $t > r^{-1}$ the oscillating part becomes small and the steady flow remains.

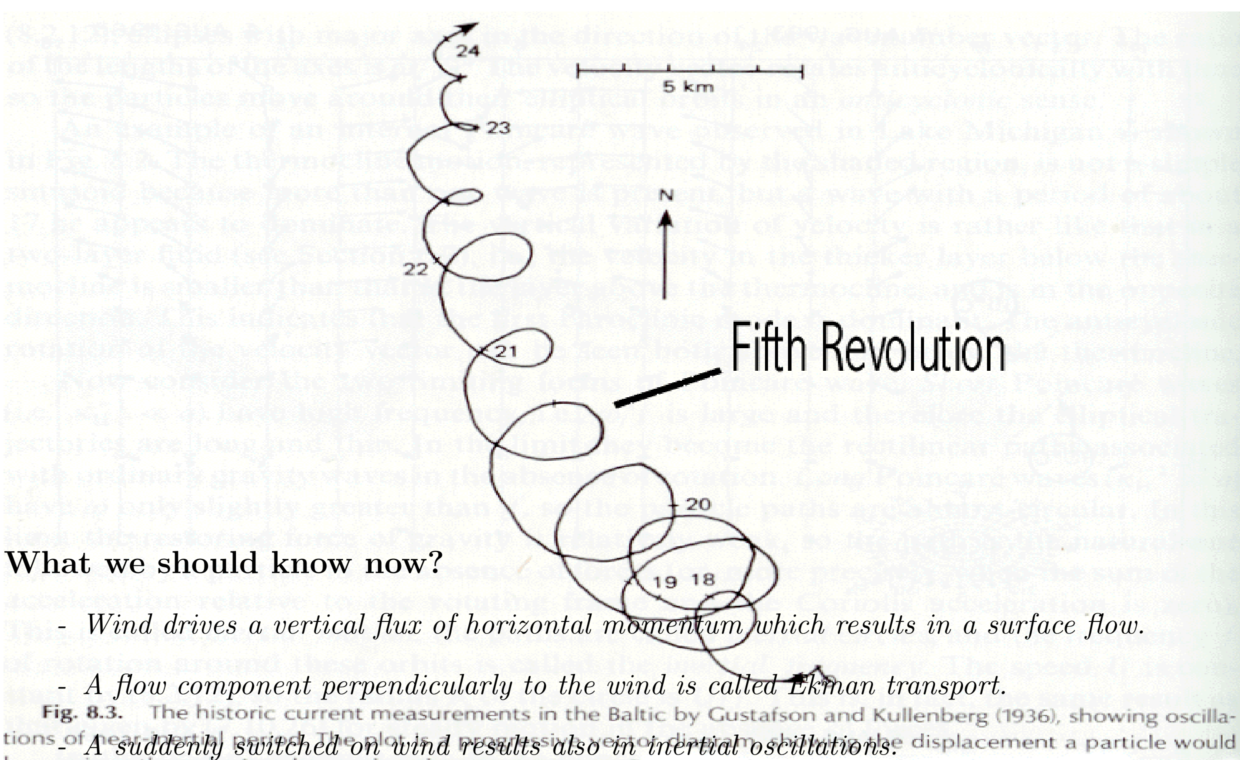


Fig. 8.3. The historic current measurements in the Baltic by Gustafson and Kullenberg (1936), showing oscillations of near-inertial period. The plot is a progressive vector diagram, showing the displacement a particle would have, given the velocity observed at the current meter. T

What we should know now?

- *Wind drives a vertical flux of horizontal momentum which results in a surface flow.*
- *A flow component perpendicularly to the wind is called Ekman transport.*
- *A suddenly switched on wind results also in inertial oscillations.*

- *For a homogeneous wind there is no pressure perturbation.*
- *The method of Green's function provides a formal solution of the hydrodynamic equations.*

Chapter 4

Oceanic waves

4.1. Wave kinematics

Waves are governed by physical processes periodic in space and time. A wave variable, ξ, can be described mathematically like

$$\xi(\mathbf{r}t) = a(\mathbf{r})\mathrm{e}^{i\chi(\mathbf{r}t)}. \tag{4.0}$$

a is called amplitude, χ is the phase of the wave. Generally, a large amount of such processes are superimposed, linearly or non-linearly.

The model "wave" is applicable only, if the phase χ is approximately a linear function of space and time. Consider a point $\mathbf{r}_0$ at time t_0, the following Taylor series is found,

$$\chi(\mathbf{r}t) \approx \chi(\mathbf{r}_0 t_0) + \frac{\partial \chi}{\partial t}(t - t_0) + (\boldsymbol{\nabla}\chi) \cdot (\mathbf{r} - \mathbf{r}_0)\,. \tag{4.0}$$

The derivatives have a physical meaning,

$$\omega = -\frac{\partial \chi}{\partial t}, \tag{4.0}$$

is called the *frequency*,

$$\mathbf{k} = \boldsymbol{\nabla}\chi. \tag{4.0}$$

is the *wave number*. Since $\mathbf{k}$ is a gradient, its curl must vanish,

$$\boldsymbol{\nabla} \times \mathbf{k} = 0. \tag{4.0}$$

Now we consider the equation

$$\chi(\mathbf{r}t) = const. \tag{4.0}$$

It defines those points $\mathbf{r}, t$ (space and time) have identical phase χ. Usually these points form lines in a plane or surfaces in space. These lines or surfaces are moving with speed $\mathbf{c}_p = \frac{d\mathbf{r}}{dt}$ defined by the equation,

$$\frac{d\chi}{dt} = \frac{\partial\chi}{\partial t} + \mathbf{c}_p \cdot \boldsymbol{\nabla}\chi = 0, \tag{4.0}$$

To be more specific we consider a *monochromatic* sine/cosine-shaped wave,

$$\xi(\mathbf{r}t) = a\mathrm{e}^{i(\mathbf{k}\cdot\mathbf{r}-\omega t}), \tag{4.0}$$

charcterised by one frequency and wave number. The condition

$$\mathbf{k} \cdot \mathbf{r} = const \tag{4.0}$$

defines planes of constant phase. The wave number vector orients perpendicularly to these planes, such waves are called plane waves. Within a peroiod dt the phase moves by $dr = \frac{\omega}{k} dt$ in direction of the wave number vector. This gives the relation

$$\mathbf{c}_p = \frac{\omega}{k}\frac{\mathbf{k}}{k} \tag{4.0}$$

between phase velocity, frequency and wave number, which is called *dispersion relation.*

If the medium is moving with a speed U some generalisation is needed. Assume that $\mathbf{r}$ is a vector in the co-ordinate system at rest (the observer is in this system), and $\mathbf{r}'$ points to the same position but in the moving system. At $t = 0$ both co-ordinate systems shall be coincide. Hence,

$$\mathbf{r} = \mathbf{r}' + \mathbf{U}t. \tag{4.0}$$

U may vary in space and time, but the scales are much larger than a wave length

$$\lambda = \frac{2\pi}{k}. \tag{4.0}$$

Considering the same point in both the co-ordinate systems, the phase is the same,

$$\chi(\mathbf{r}t) = \bar{\chi}(\mathbf{r}'t) = \bar{\chi}(\mathbf{r} - \mathbf{U}t, t). \tag{4.0}$$

The derivatives to $\mathbf{r}$ and $\mathbf{r}'$, i.e. the wave number is the same in both co-ordinate systems, but the frequency differs,

$$\begin{aligned} \omega &= -\frac{\partial\chi}{\partial t} \\ &= -\left.\frac{\partial\bar{\chi}}{\partial t}\right|_{\mathbf{r}'} + \mathbf{U} \cdot \boldsymbol{\nabla}\chi \\ &= \sigma + \mathbf{k} \cdot \mathbf{U}. \end{aligned} \tag{4.-1}$$

σ isw called the *intrinsic frequency* of the wave process within a resting medium. Cross-differentiation gives

$$\frac{\partial \mathbf{k}}{\partial t} + \boldsymbol{\nabla}\left(\sigma + \mathbf{k}\cdot\mathbf{U}\right) = 0. \tag{4.-1}$$

The generalised phase velocity is

$$\mathbf{c}_p = \left(\frac{\sigma + \mathbf{U}\cdot\mathbf{k}}{k}\right)\frac{\mathbf{k}}{k}, \tag{4.-1}$$

the phase changes not only due to wave spreading but also from the moving medium.

In the more general case, the intrinsic frequency is not a constant but depends on the wave number,

$$\sigma = \sigma(\mathbf{k}). \tag{4.-1}$$

This dispersion relation cannot be found from wave kinematics but is subject of wave dynamics. We will see later, that the knowledge of the dispersion relation of oceanic waves is fundamental for its theory.

The wave number as well as the frequency may depend on time. Cross differentiation gives

$$\frac{\partial \mathbf{k}}{\partial t} + \boldsymbol{\nabla}\omega = 0. \tag{4.-1}$$

From the dispersion relation it follows for the time development of the wave number,

$$\frac{\partial \mathbf{k}}{\partial t} + \left(\mathbf{U} + \frac{\partial \sigma}{\partial \mathbf{k}}\right)\cdot\boldsymbol{\nabla}\mathbf{k} = 0. \tag{4.-1}$$

This is similar to an equation of continuity. The quantity

$$\mathbf{c}_g = \frac{\partial \sigma}{\partial \mathbf{k}} \tag{4.-1}$$

is called *group velocity*. It describes the flow of the *phase density*. The wave number is the number of surfaces with equal phase per unit length and is in this sense a density. Generally $\mathbf{c}_g$ differs from the phase velocity, it may have even the opposite direction. All wave types, where the phase and the group velocity are called *dispersive*.

To show the physical meaning, we consider a "group" of waves with similar wave number. The distribution should be bell shaped with a maximum at $k = k_0$,

$$a(k) = A\mathrm{e}^{-\frac{(k-k_0)^2}{2\kappa^2}}. \tag{4.-1}$$

The amplitude can be normalises like

$$A = \frac{1}{\kappa\sqrt{2\pi}}. \tag{4.-1}$$

A physical quantity $\xi(xt)$ is the superposition by many mpatial waves with amplitudes $a(k)$,

$$\xi(xt) = \int_{-\infty}^{\infty} dk a(k) \mathrm{e}^{i(kx-\omega(k)t)} \tag{4.-1}$$

Notable, the integral can be carried out only if the dispersion relation is known.

Approximating the wave group be its properties at $k = k_0$, the frequency near this points determines $\xi(xt)$,

$$\begin{aligned} \omega(k) &\approx \omega(k_0) + \left.\frac{\partial\omega}{\partial k}\right|_{k=k_0} (k-k_0) + \left.\frac{\partial^2\omega}{\partial k^2}\right|_{k=k_0} \frac{(k-k_0)^2}{2} + ... \\ &\approx Uk + \sigma(k_0) + c_g(k_0)(k-k_0) + ... \end{aligned} \tag{4.-1}$$

Here

$$c_g(k) = \frac{\partial\sigma}{\partial k} \tag{4.-1}$$

is the one-dimensional group velocity.
With this approximation the integral can be carried out,

$$\xi(xt) = \xi(x - Ut - c_g t, 0)\mathrm{e}^{i(c_g k_0 - \sigma(k_0))t}. \tag{4.-1}$$

The wave group keeps its form and moves with the velocity $U + c_g$. For large time higher order terms need consideration. This usually describes disintegration of the wave group.

The energy of waves consistes of kinetic and potential energy. According to the virial-theorem, both contribute with the same amount. Hence, the total energy must be proportional to the square of the elevation in the earth's gravitation field,

$$E(xt) = |\xi(xt)|^2 = \int dk \int dk' a(k)a(k')\mathrm{e}^{i((k-k')x-(\omega(k)-\omega(k'))t)}. \tag{4.-1}$$

For a bell shaped wave group like the aforementioned Taylor expansion is possible

$$\omega(k) - \omega(k') \approx (U + c_g(k_0))(k - k'). \tag{4.-1}$$

The corresponding energy is

$$E(xt) \approx \int dk \int dk' a(k)a(k')\mathrm{e}^{i(k-k')(x-(U+c_g)t)}. \tag{4.-1}$$

and the flux reads

$$\frac{\partial E}{\partial t} + (U + c_g)\frac{\partial E}{\partial x} = 0. \tag{4.-1}$$

The group velocity determines the energy flux in relation the wave spreading.

4.2. Surface gravity waves in the non-rotating ocean

4.2.1. The deep water case

We consider waves at the surface of a deep model ocean. The water should be inviscid and incompressible, the density ρ is approximately constant. Incompressibility implies infinite sound velocity, a pressure perturbation spreads infinitely fast. We will see below that this approximation is well justified, all wave velocities are much smaller than the sound velocity $\approx 1500\text{m s}^{-1}$. For an incompressible fluid the condition

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0, \tag{4.-1}$$

is valid. Surface waves are know to spread with lttle damping over long distances. This justifies to neglect the shear part of the stress tensor.

For the beginning we consider small amplitudes and can study the linearised hydrodynamic equations. This excludes for the moment non-linear wave phenomena like wave-wave interaction or solitons.

What obout the Coriolis force? We consider time derivatives and Coriolis forces of the orders

$$\frac{U}{T}, \quad \text{and} \quad fU \tag{4.0}$$

Keeping in mind, that f is of the order $10^{-4}\,\text{s}^{-1}$ the influence of the Coriolis term should be small.

The momentum equations become simplified to

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \rho \mathbf{g} z, \tag{4.1}$$

the fluid is accelerated due to pressure gradients and gravity.

We define the coordinate System in such a way, that the origin is at the surface of the fluid at rest. z is positive upwards and negative downwards. $\mathbf{g}$ is directed downwards and counts negative. For a fluid at rest the pressure p consists of the atmosphere pressure and the hydrostatic pressure,

$$\begin{aligned} p &= p_a + p_0, \\ p_0 &= -\rho g z. \end{aligned} \tag{4.1}$$

If the sea surface is elevated, there exists an additional small excess pressure p_e and the total pressure is

$$p = p_a + p_0 + pe. \tag{4.2}$$

The air pressure varies at large scales only. However, from wind blowing over wave crests, local pressure maxima and minima are generated. These are essential for the wind - wave interaction. This is still to complex to understand the spreading of surface waves. So we restrict ourselves to free waves and assume a constant air pressure for the moment.

The remaining momentum equation reads now,

$$\rho \frac{\partial \mathbf{u}}{\partial t} = -\nabla p_e, \tag{4.3}$$

the hydrostatic pressure component balances gravity.

The curl of the momentum equation reads

$$\rho \frac{\partial \boldsymbol{\nabla} \times \mathbf{u}}{\partial t} = 0, \tag{4.4}$$

a current field without a curl remains free of rotation and is described by a potential Φ. The velocity components are given by the gradient of the potential

$$\mathbf{u} = \boldsymbol{\nabla}\Phi. \tag{4.5}$$

From the continuity equation we find that the potential Φ is solution of a Laplace equation

$$\Delta\Phi = 0, \tag{4.6}$$

where Δ is the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{4.7}$$

A Laplace equation with fixed boundary conditions does not describe any wave phenomena. Hence, the key to find a wave-like solution is the surface boundary condition. The sea surface elevation is described by a function

$$\xi(x, y, t) - z = 0. \tag{4.8}$$

Applying the operator $\frac{D}{Dt}$ relates the vertical velocity to the time derivative of ξ,

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} - w = 0. \tag{4.9}$$

For consistency with the momentum equations the non-linear terms are neglected,

$$\frac{\partial \xi}{\partial t} = w = \frac{\partial \Phi}{\partial z} \quad \text{for} \quad z = \xi, \tag{4.10}$$

spreading of an elevation by currents is neglected.

The surface pressure is the air pressure,

$$p = p_a \quad \text{for} \quad z = \xi, \tag{4.11}$$

hence the pressure perturbation p_e must be opposite to the hydrostatic pressure

$$p_e = -p_0 = \rho g \xi \quad \text{for} \quad z = \xi. \tag{4.12}$$

We eliminate now the pressure and ξ from the surface boundary condition and try to express these quantities in terms of the potential Φ. To this end we integrate the vertical momentum equation vertically under the assumption, that Φ as well as p_e vanish in large depth,

$$\Phi = 0, \quad p_e = 0 \quad \text{for} \quad z \to -\infty. \tag{4.13}$$

We find

$$p_e = -\rho \frac{\partial \Phi}{\partial t} \quad \text{for} \quad z = \xi. \tag{4.14}$$

Now we express p_e by means of the boundary condition (4.12) by ξ. For the time derivative of ξ boundary condition 4.10 helps to eliminate ξ completely:

$$\frac{\partial^2 \Phi}{\partial t^2} = -g \frac{\partial \Phi}{\partial z} \quad \text{for} \quad z = \xi. \tag{4.15}$$

This is still a complex expression. But now small surfa ce elevations can be assumed and a Tayler expansion near $z = 0$ is possible. (Note, the convergence can be justified later from the solution of the resulting differential equation. The wave height must be much smaller than the wave length.) The first order reads

$$\frac{\partial^2 \Phi}{\partial t^2} = -g \frac{\partial \Phi}{\partial z} \quad \text{for} \quad z = 0. \tag{4.16}$$

we will see that this is a wave equation.

To solve the Laplace equation for the wave potential we introduce a Fourier decomposition,

$$\Phi(x, y, z, t) = \Psi(z) \mathrm{e}^{i(\omega t - k_x x - k_y y)}. \tag{4.17}$$

Insertion in the Laplace equation gives an equation for $\Psi(z)$,

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial z^2} - k^2 \Psi &= 0, \\ k^2 &= k_x^2 + k_y^2. \end{aligned} \tag{4.17}$$

Insertion into the surface boundary condition 4.16 gives a boundary condition for Ψ,

$$\omega^2 \Psi = g \frac{\partial \Psi}{\partial z} \quad \text{for} \quad z = 0. \tag{4.18}$$

The partial equation for Ψ is of second order. The second boundary condition applies at $z \to -\infty$,

$$\Psi = 0 \quad \text{for} \quad z \to -\infty. \tag{4.19}$$

The differential equation for Ψ together with the boundary conditions for an eigen value problem that delivers the solution for Ψ but also the possible values for frequency and wave numbers, the *frequency spectrum*. The solution is

$$\Psi(z) = \Psi_0 \mathrm{e}^{kz}. \tag{4.20}$$

(Note, z is negative!) The second solution with positive exponential vanishs because of the boundary condition at $(z \to -\infty)$.

From the surface boundary condition we find a relation between frequency and wave number

$$\omega^2 = gk. \tag{4.21}$$

The phase velocity is

$$c_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} = \frac{g}{\omega}. \tag{4.22}$$

The phase velocity depends on frequency or wave number respectively. Long waves or waves with low frequency move faster than short waves with high wave frequency. A wave packet composed from waves with different frequency is decomposing. Such waves are called *dispersive*. Finally, it is easy to check, that waves with wave length $\lambda < 100$m are slow enough to fulfill the condition $c_p \ll c_{sound}$. The application of the incompressibility condition is well justified.

Now we consider the flow field. We turn the coordinate system so that the wave direction (the direction of (k_x, k_y)) is the x-direction. The (real part) of the velocity field is

$$u = \mathrm{Re}\frac{\partial \Phi}{\partial x} = k\Psi_0 \mathrm{e}^{kz} \sin(\omega t - kx) \tag{4.23}$$

$$w = \mathrm{Re}\frac{\partial \Phi}{\partial z} = k\Psi_0 \mathrm{e}^{kz} \cos(\omega t - kx). \tag{4.24}$$

The velocity amplitude is

$$|\mathbf{u}| = k\Psi_0 \mathrm{e}^{kz}, \tag{4.25}$$

The direction rotates clockwise in the x-z-plane with the frequency ω and varies horizontally with wave number k. The phase is independent of depth, but the amplitude decreases exponentielly with depth. The typical penetration depth is the wave length.

Trajectories of fluid elements can be found as solution of the differential equations

$$\frac{dx}{dt} = u(x(t), t), \tag{4.26}$$

$$\frac{dz}{dt} = w(x(t), t). \tag{4.27}$$

For small amplitudes the solution is approximately

$$x(t) = x_0 - \frac{k}{\omega}\Psi_0 \mathrm{e}^{kz}\cos(\omega t - kx), \tag{4.28}$$

$$z(t) = z_0 - \frac{k}{\omega}\Psi_0 \mathrm{e}^{kz}\sin(\omega t - kx), \tag{4.29}$$

hence, trajectories are approximately circles with a radius

$$\frac{\Psi_0}{c_p}\mathrm{e}^{kz}. \tag{4.30}$$

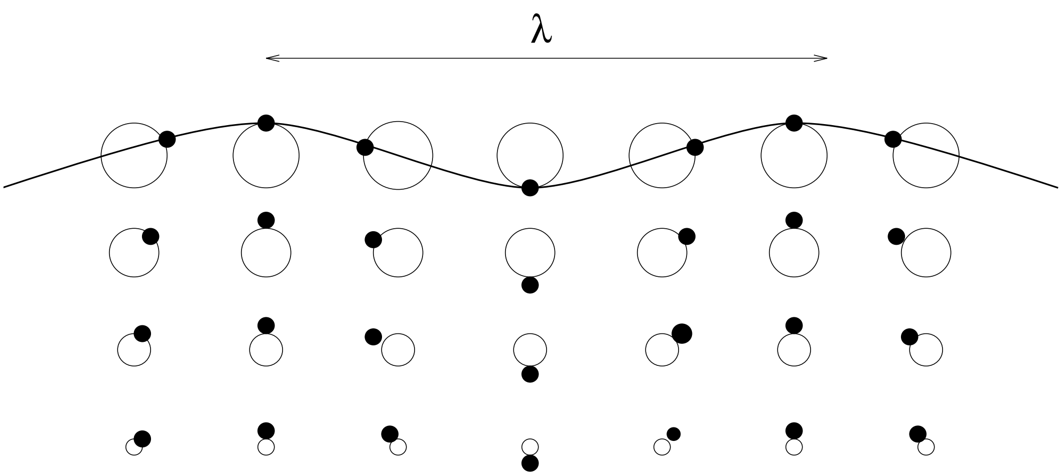


Figure 4.1. Orbital movement of fluid elements in relation to a monochromatic surface gravity wave.

Figure 4.1 shows trajectories for a monochromatic surface wave.

4.2.2. The influence of the bottom

Now we relax the restriction to an infinitely deep ocean. All consideration may be repeated, but now with the bottom boundary condition of a vanishing mass flux through the bottom

$$w = \frac{\partial\Phi}{\partial z} = \frac{\partial\Psi}{\partial z} = 0 \quad \text{for} \quad z \rightarrow -H. \tag{4.31}$$

The solution for $\Psi(z)$ contains now also the exponentially increasing part

$$\Psi(z) = \Psi_1 \mathrm{e}^{kz} + \Psi_2 \mathrm{e}^{-kz}. \tag{4.32}$$

Surface and bottom boundary condition give two equations for Ψ_1 and Ψ_2,

$$\begin{aligned} \left(\omega^2 - gk\right)\Psi_1 + \left(\omega^2 + gk\right)\Psi_2 &= 0, \\ k\left(\Psi_1 \mathrm{e}^{-kH} - \Psi_2 \mathrm{e}^{kH}\right) &= 0. \end{aligned} \tag{4.32}$$

For a solution exists the coefficient determinant must vanish, which results in the relation between frequency and wave number

$$\omega^2 = gk\tanh(kH). \tag{4.33}$$

For Ψ and the velocities we find

$$\Psi(z) = \Psi_0\,\mathrm{e}^{-kH}\cosh\left(k(z+H)\right), \tag{4.34}$$

$$u = \mathrm{Re}\frac{\partial\Phi}{\partial x} = k\Psi_0\,\mathrm{e}^{-kH}\cosh\left(k(z+H)\right)\sin(\omega t - kx) \tag{4.35}$$

$$w = \mathrm{Re}\frac{\partial\Phi}{\partial z} = k\Psi_0\,\mathrm{e}^{-kH}\sinh\left(k(z+H)\right)\cos(\omega t - kx). \tag{4.36}$$

We discuss now the limiting cases

- $kH \gg 1$, i.e., deep water,
- $kH \ll 1$, i.e., shallow water

"Deep" and "shallow" are meant in relation to a horizontal scale, namely to the wave length $\lambda = \frac{2\pi}{k}$. The deep water case is discussed previously, for $H \to \infty$ we retain the equations derived in the previous section. For $H \to 0$ the shallow water solution

$$\omega^2 \approx gHk^2\left(1 - \frac{1}{3}k^2H^2\right) \tag{4.37}$$

$$\Psi(z) \approx \Psi_0\,\mathrm{e}^{-kH}\left(1 + \frac{(z+H)^2k^2}{2}\right). \tag{4.38}$$

Bottom and surface boundary condition are still fulfilled, but the solution becomes independent of z, the elongation becomes mostly horizontal. We will return to this point, when the "shallow water equations" will be discussed in more detail.

The z-dependency of the velocity amplitudes is different for u and w. Hence, the circles for the deep water case are replaced by elliptical trajectories as shown in Figure 4.2. In the "shallow water" case the velocities are

$$u = \mathrm{Re}\frac{\partial\Phi}{\partial x} \approx k\Psi_0\,\mathrm{e}^{-kH}\left(1 + \frac{(z+H)^2k^2}{2}\right)\sin(\omega t - kx) \tag{4.39}$$

$$w = \mathrm{Re}\frac{\partial\Phi}{\partial z} \approx k\Psi_0\,\mathrm{e}^{-kH}\,k\,(z+H)\cos(\omega t - kx). \tag{4.40}$$

Finally the sea surface elevation can be determined by time integration of the surface

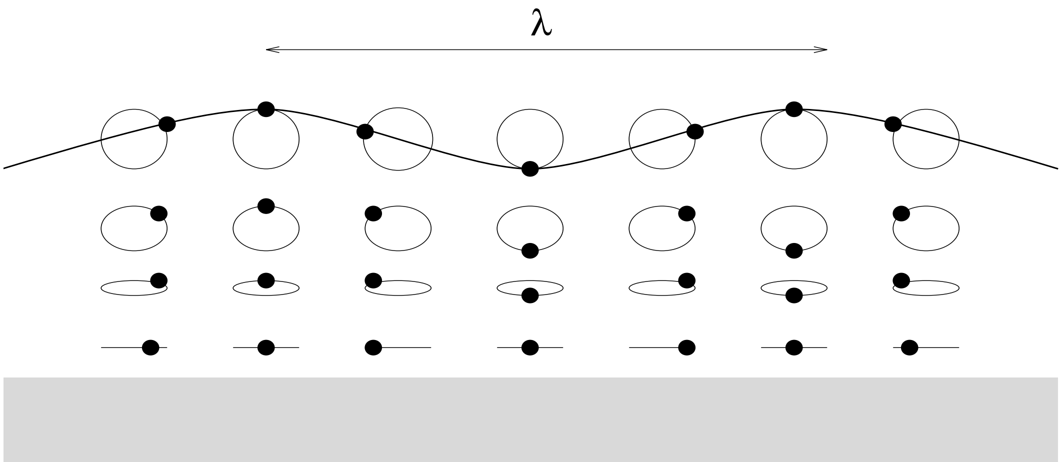


Figure 4.2. Orbital movement of fluid elements in relation to a monochromatic surface gravity wave for finite depth.

boundary condition

$$\begin{aligned} \xi(t) &= \xi(0) + \int_0^t dt' w(t, z=0) \\ &= \xi(0) + \frac{k}{\omega}\Psi_0 \, e^{-kH} \sinh(kH) \left(\sin(\omega t - kx) + \sin(kx)\right). \end{aligned} \tag{4.39}$$

4.2.3. The Stokes drift

The trajectories found for the orbital wave motion are rough approximations to demonstrate basic principles. A more thourough analysis reveals that the orbital speed at the surface is higher than in the deeper layers. Hence, a fluid elements will not return exactly to its starting point but will end up a little bit advanced in the direction of the wave spreading. This can be related to an average momentum of the fluid gained from the wave field, the so called *Stokes drift*.

Our Eulerian equations are linearised and do not reveal the Stokes drift. However, the velocity field shows the vertical gradient of the horizontal amplitude and a Lagrangian analysis of the resulting trajectories should be suitable to understaned this effect.

We track a particale starting at $t = 0$ at the point $\mathbf{a} = (x_0, z_0)$. At time t it will arrive at point

$$\mathbf{x} = \mathbf{a} + \int_0^t dt' \mathbf{u}^L(\mathbf{a}, t') \tag{4.40}$$

The starting point a is a marker for the fluid element. The Lagrangian velocity at this point is $\mathbf{u}^L(\mathbf{a},t)$, the Stokes drift may be defined as an average over one wave period,

$$\bar{\mathbf{u}}^L = \frac{1}{T}\int_0^T dt'\mathbf{u}^L(\mathbf{a},t'). \tag{4.41}$$

Now we express the unknown Lagrangian velocity in terms of the Eulerian velocity. At time t at point x both are equal,

$$\mathbf{u}^L(\mathbf{a},t) = \mathbf{u}^E(\mathbf{x}(t),t). \tag{4.42}$$

Averaging over one wave period without an exact consideration of the time dependency of the position vector x, the average value vanishes, $\bar{\mathbf{u}}^L = 0$, trajectories are periodic. This is the approximation used previously. Now we consider the time dependency of x and find,

$$\mathbf{u}^L(\mathbf{a},t) = \mathbf{u}^E\left(\mathbf{a}+\int_0^t dt'\mathbf{u}^L(\mathbf{a},t'),t\right). \tag{4.43}$$

With a Taylor expansion around $\mathbf{x}=\mathbf{a}$ we find a correction,

$$\mathbf{u}^L(\mathbf{a},t) \approx \mathbf{u}^E(\mathbf{a},t)+\left(\int_0^t dt'\mathbf{u}^E(\mathbf{a},t')\right)\cdot\boldsymbol{\nabla}\mathbf{u}^E(\mathbf{a},t)+... \tag{4.44}$$

The time average of the first term vanishes, the average of the second term is

$$\bar{u}^L = \Psi_0^2k^2\mathrm{e}^{-2kH}\frac{k}{2\omega}\cosh\left(2k(z+H)\right), \tag{4.45}$$

$$\bar{w}^L = 0. \tag{4.46}$$

With equation (4.39) the amplitude of the surface elevation can be introduced,

$$\xi_0 = \psi_0\frac{k}{\omega}\mathrm{e}^{-kH}\sinh(kH). \tag{4.47}$$

and the horizontal Stokes drift is approximately

$$\bar{u}^L = \xi_0^2k^2\frac{c}{2}\frac{\cosh\left(2k(z+H)\right)}{\sinh^2(kH)}. \tag{4.48}$$

The vertical component is zero. The Stokes drift is directed into the direction of the waves phase velocity. The order of magnitude is some cm/s. It may be enhanced if bopttom and bottom friction reduce the orbital velocity near the bottom. For long waves in the deep water it is generally small.

Stokes drift is a non-linear effect. As such it is not to be seen in the Eulerian solution of the linearised equations. The Lagrangian flow calculated here is a perturbational result and considers the lowest order of the advection of waves by the wave field itself.

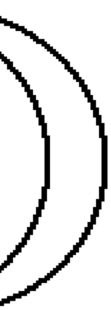
Figure 4.3. Orbital movement of fluid elements with consideration of the Stokes drift.

4.2.4. Dispersion of surface gravity waves

Between frequency and wave number of surface gravity waves a very general result could be derived,

$$\omega^2(k) \quad = \quad gk \tanh kH. \tag{4.49}$$

This result applies for all waves with inviscid dynamics governed by inertial and gravity forces. Only for very short waves surface tension comes into plays and a more general consideration is needed.



This dispersion relation defines two velocities:

- the phase speed $\mathbf{c}_p = \frac{\omega}{k}\frac{\mathbf{k}}{k}$ for the spreading of surfaces with equal phase
- the group velocity $\mathbf{c}_g = \frac{\partial \omega}{\partial \mathbf{k}}$ for the spreading of areas with constant wave number as well as energy spreading with waves.

The group velocity is

$$\mathbf{c}_g \quad = \quad \frac{\mathbf{c}_p}{2}\left(1 + \frac{2kH}{\sinh 2kH}\right). \tag{4.50}$$

for deep water, $kH \gg 1$, the group velocity is half the phase speed, in the shallow water case we find $c_g = c_p = c$ and the dispersion becomes zero. Surface gravity waves show normal dispersion, capilarry waves may shoe anomal dispersion.

4.2.5. The energy budget of waves

Now we consider the potential energy of a fluid element in the water body, when a sin-shaped wave is spreading. Elevation by $\delta = \int_0^t dt' w(t)$ is related to a potential energy

$$e_{pot} \quad = \quad \rho g \delta. \tag{4.51}$$

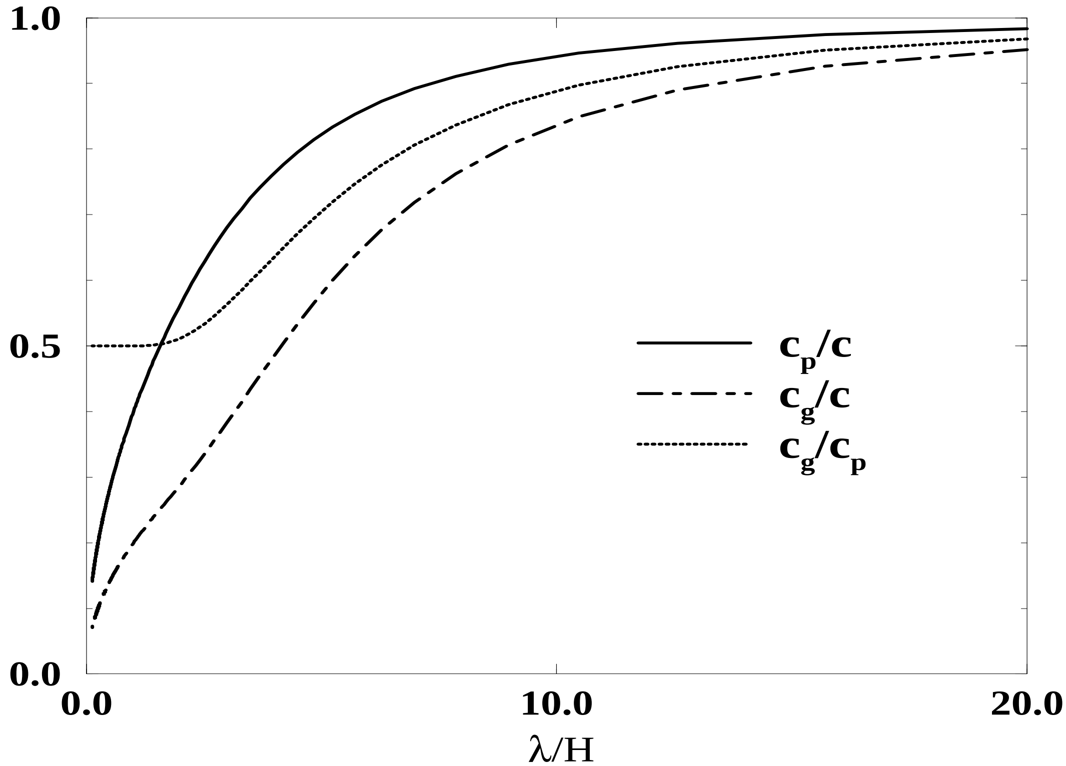


Figure 4.4. Phase and group velocity of surface gravity waves as function of the wave length. The ratio to the shallow water approximation $c_p = c_g = c = \sqrt{gH}$ is shown.

Integration over all fluid elements from $z = 0$ to the surface $z = \xi$ gives

$$\int_0^\xi dz e_{pot}(z) = \rho g \frac{\xi^2}{2}, \tag{4.52}$$

integration over one wave length gives the potantial energr within one wave length,

$$\begin{aligned} E_{pot} &= \int_0^\lambda dx \int_0^\xi dz e_{pot} \\ &= \rho g \frac{\lambda}{2} \frac{k^2}{\omega^2} \Psi_0^2 \mathrm{e}^{-2kH} \sinh^2 kH \\ &= \frac{\pi}{2} A^2 \tanh kH. \end{aligned} \tag{4.51}$$

To find the last equation, the dispersion relation was used. A isthe amplitude of the velocity potential,

$$A = \Psi_0 \mathrm{e}^{-kH} \cosh kH. \tag{4.52}$$

We consider a harmonic oscillation and the Virial-theorem must be valid. Hence, the total energy is

$$E = 2E_{pot} = \pi A^2 \tanh kH. \tag{4.53}$$

Now we consider the work to be done by a horizontqal displacement of a fluid element against the pressure p_e. (all contributions of the hydrostatic pressure cancel out.) The movement happens within the time span dt by $dx = udt$, i.e., the workm is

$$dW = p_e u dt. \tag{4.54}$$

Recall

$$p_e = -\rho \frac{\partial \Phi}{\partial t} \tag{4.55}$$

and

$$u = \frac{\partial \Phi}{\partial x}. \tag{4.56}$$

The work carried out during a wave period T within the full water column is

$$W = -\int_0^T dt \int_{-H}^0 dz \rho \frac{\partial \Phi}{\partial t} \frac{\partial \Phi}{\partial x}. \tag{4.57}$$

Performing the integrals results in

$$\begin{aligned} W &= \frac{\pi}{2} A^2 \tanh kH \left(1 + \frac{2kH}{\sinh 2kH}\right) \\ &= E \frac{c_g}{c_p}. \end{aligned} \tag{4.57}$$

With $c_p = \frac{\lambda}{T}$ this can be written in the form

$$\frac{W}{T} = c_g \frac{E}{\lambda}. \tag{4.58}$$

The left hand side expression is the horizontal energy flux through a vertical plane during one wave period. The right hand side is the flux of energy within one wave length with group velocity c_g. We have confirmed, that the work done by horizontal flow agains the pressure field corresponds to an energy flux with group velocity.

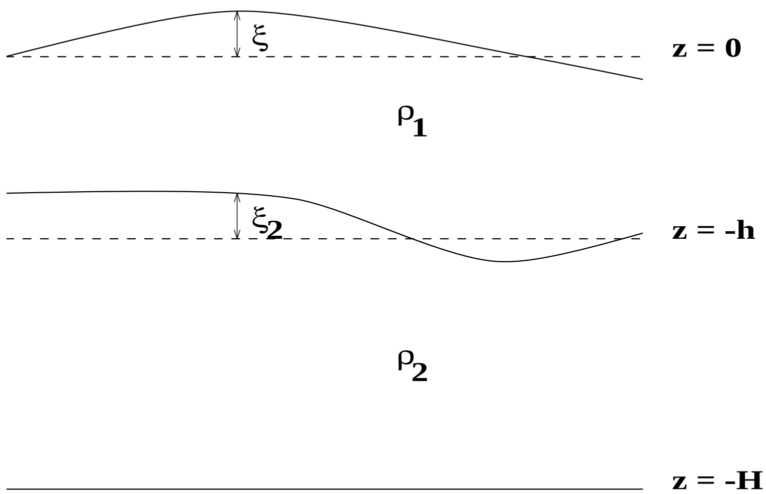


Figure 4.5. Geschichtetes Gewässer mit zwei überlagerten Wassermassen verschiedener Dichte.

Interne Schwerewellen

Die Ausbreitung interner Wellen gehört zu den schwierigen Aufgaben der theoretischen Mechanik. In diesem Abschnitt soll nur diskutiert werden, wie sich das Wellenspektrum in einem geschichteten im Vergleich zum ungeschichteten Wasserkörper verändert. Für eine detaillierte Diskussion sei auf die Literatur verwiesen.

Wir betrachten ein Gewässer, das aus zwei Wassermassen aufgebaut ist, die sich in Temperatur oder Salzgehalt wesentlich unterscheiden. Das kann z.B. ein Fjord sein, in dem aus Flüssen stammendes Süßwasser mit Ozeanwasser überlagert ist. Wir können aber auch einen See, oder auch ein Meer betrachten, in dem sich in ruhiger sommerlicher Wetterlage eine thermische Sprungschicht herausgebildet hat. Hier ist lediglich entscheidend, daß sich die Dichte an der Sprungschicht ändert. Die Betrachtung einer kontinuierlichen Schichtung ist komplizierter und wird später für Wellen im rotierenden Ozean besprochen.

Anstatt eine tiefenabhängige Dichte einzuführen, teilen wir die Wassermasse in zwei Horizonte mit den Dichten ρ_1 und ρ_2 auf. Alle übrigen Größen werden, wenn nötig, ebenfalls indiziert.

Wie für den homogenen Ozean vernachlässigen wir Reibung, nichtlineare Effekte sowie äußere Kräfte. Die Bewegungsgleichung ist dann die linearisierte Euler'sche Gleichung

$$\rho_i \frac{\partial \mathbf{u}_i}{\partial t} = -\boldsymbol{\nabla} p_i + \rho_i g. \tag{4.58}$$

Aus der Rotationsfreiheit des Geschwindigkeitsfeldes folgt die Potentialdarstellung

$$\mathbf{u}_i = \boldsymbol{\nabla} \Phi_i, \tag{4.58}$$

wobei aus der Kontinuitätsgleichung eine Laplacegleichung für das Potential folgt,

$$\Delta\Phi_i = 0. \tag{4.58}$$

Der Druck kann wiederum in einen atmoshphärischen Anteil, einen hydrostatischen Gleichgewichtsanteil und eine Abweichung von Gleichgewicht aufgeteilt werden,

$$p_i = p_a + p_{i0} + p_{ie}. \tag{4.58}$$

Der Gleichgewichtsdruck kann durch Intergation der Gleichgewichtsbedingung $\boldsymbol{\nabla} p_{i0} - \rho_i \mathbf{g} = 0$ gewonnen werden:

$$\begin{aligned} p_{10} &= -\rho_1 g z, && (4.59)\\ p_{20} &= \rho_1 g h - \rho_2 g(z+h) = gh(\rho_1 - \rho_2) - \rho_2 g z. && (4.60) \end{aligned}$$

Damit kann wiederum die Schwerkraft aus den Euler'schen Gleichungen eliminiert werden,

$$\rho_i \frac{\partial \mathbf{u}_i}{\partial t} = -\boldsymbol{\nabla} p_{ie}. \tag{4.60}$$

Es müssen Randbedingungen an den Grenzschichten, d.h. an der Oberfläche, an der Grenzschicht zwischen den Wassermassen und am Boden formuliert werden. Für die w-Komponente finden wir, daß die Geschwindigkeit der Grenzschichten identisch mit w an selbigen sein muß,

$$\begin{aligned} w_1 &= \frac{\partial \xi}{\partial t} \qquad z = \xi, && (4.61)\\ w_1 &= \frac{\partial \xi_2}{\partial t} \qquad z = -h + \xi_2, && (4.62)\\ w_1 &= w_2 \qquad z = -h + \xi_2, && (4.63)\\ w_2 &= 0 \qquad z = -H. && (4.64) \end{aligned}$$

Der Druck an der Oberfläche ist der atmospärische Druck, der wiederum als konstant vorausgesetzt wird. Daraus folgt sofort

$$p_{e1} = -p_{01} = \rho_1 g \xi \qquad z = \xi. \tag{4.64}$$

An der Dichtegrenzschicht muß der Druck stetig sein. Mit dem obigen Ausdruck für p_0 folgt

$$\begin{aligned} p_{e2} &= p_{e1} + p_{01} - p_{02}\\ &= p_{e1} + (\rho_2 - \rho_1)\, g \xi_2 \qquad z = -h + \xi_2. && (4.64) \end{aligned}$$

Nun sollen alle Größen in den Randbedingungen durch das Potential Φ ausgedrückt werden. Dazu ersetzen wir in den Euler'schen Gleichungen die Geschwindigkeiten durch die Gradienten des Potentials Φ und integrieren die vertikale Eulersche Gleichung,

$$\rho_1 \frac{\partial \Phi_1}{\partial t} = -p_{1e} + f_1(x), \tag{4.65}$$

$$\rho_2 \frac{\partial \Phi_2}{\partial t} = -p_{2e} + f_2(x). \tag{4.66}$$

Durch Einsetzen in die horizontale Gleichung kann man zeigen, daß f_1 und f_2 höchstens konstant sein können und damit dynamisch nicht relevant sind. Mit obigen Gleichungen und den Randbedingen folgen die Randbedingungen für das Potential Φ

$$\rho_1 \frac{\partial^2 \Phi_1}{\partial t^2} = -\rho_1 g \frac{\partial \Phi_1}{\partial z} \qquad z = \xi, \tag{4.67}$$

$$\rho_2 \frac{\partial^2 \Phi_2}{\partial t^2} - \rho_1 \frac{\partial^2 \Phi_1}{\partial t^2} = -\left(\rho_2 - \rho_1\right) g \frac{\partial \Phi_2}{\partial z} \qquad z = -h + \xi_2, \tag{4.68}$$

$$\frac{\partial \Phi_2}{\partial z} = \frac{\partial \Phi_2}{\partial z} \qquad z = -h + \xi_2, \tag{4.69}$$

$$\frac{\partial \Phi_2}{\partial z} = 0 \qquad z = -H. \tag{4.70}$$

Mit dem Lösungsansatz

$$\Phi_k = \Psi_k(z) \mathrm{e}^{i(\omega t - kx)} \tag{4.70}$$

können horizontale, vertikale und Zeitkoordinate separiert werden. Aus der Potentialgleichung folgt eine Gleichung für Ψ,

$$\frac{\partial^2 \Psi_k}{\partial z^2} - k^2 \Psi_k = 0, \tag{4.70}$$

die Randbedingung erhalten die Form

$$\omega^2 \Psi_1 = g \frac{\partial \Psi_1}{\partial z} \qquad z = \xi, \tag{4.71}$$

$$\omega^2 \left(\rho_2 \Psi_2 - \rho_1 \Psi_1\right) = \Delta\rho g \frac{\partial \Psi_2}{\partial z} \qquad z = -h + \xi_2, \tag{4.72}$$

$$\frac{\partial \Psi_1}{\partial z} = \frac{\partial \Psi_2}{\partial z} \qquad z = -h + \xi_2, \tag{4.73}$$

$$\frac{\partial \Psi_2}{\partial z} = 0 \qquad z = -H. \tag{4.74}$$

$\Delta\rho$ steht abkürzend für $\rho_2 - \rho_1$.

Wie im homogenen Fall kann dieses Gleichungssystem mit einem Exponentialansatz gelöst werden,

$$\Psi_1(z) = A\mathrm{e}^{-kz} + B\mathrm{e}^{kz}, \quad (4.75)$$
$$\Psi_2(z) = C\mathrm{e}^{-kz} + D\mathrm{e}^{kz}. \quad (4.76)$$

Die Koeffizienten folgen aus den Randbedingungen. Um die entstehenden Gleichungen lösbar zu halten, werden diese wiederum für kleine Elongationen um die Punkte $z = 0$ bzw. $z = -h$ entwickelt. Damit folgt das Gleichungssystem

$$A(\omega^2 + gk) + B(\omega^2 - gk) = 0, \quad (4.77)$$
$$\omega^2 \left(\rho_2 \left(C\mathrm{e}^{kh} + D\mathrm{e}^{-kh}\right) - \rho_1 \left(A\mathrm{e}^{kh} + B\mathrm{e}^{-kh}\right)\right) \quad (4.78)$$
$$+\Delta\rho g k \left(A\mathrm{e}^{kh} - B\mathrm{e}^{-kh}\right) = 0, \quad (4.79)$$
$$A\mathrm{e}^{kh} - B\mathrm{e}^{-kh} - C\mathrm{e}^{kh} + D\mathrm{e}^{-kh} = 0, \quad (4.80)$$
$$C\mathrm{e}^{kH} - D\mathrm{e}^{-kH} = 0. \quad (4.81)$$

Das ist ein homogenes Gleichungssystem, das nur dann lösbar ist, wenn die Koeffizientendeterminante verschwindet. Um die weitere Rechnung einfach zu halten, beschränken wir uns nachfolgend auf den Fall sehr großer Tiefen, $H \to \infty$. Dann muß C verschwinden und D kann durch A und B ausgedrückt werden:

$$D\mathrm{e}^{-kh} = B\mathrm{e}^{-kh} - A\mathrm{e}^{kh}. \quad (4.81)$$

Damit wird das Gleichungssystem auf 2 Gleichungen reduziert,

$$A(\omega^2 + gk) + B(\omega^2 - gk) = 0, \quad (4.82)$$
$$A\mathrm{e}^{kh} \left(-(\rho_1 + \rho_2)\omega^2 + \Delta\rho g k\right) + B\mathrm{e}^{-kh}\Delta\rho(\omega^2 - gk) = 0. \quad (4.83)$$

Die Koeffizientendeterminante verschwindet, wenn die Bedingung

$$(\omega^2 - gk)\mathrm{e}^{kh} \left((\rho_1 + \rho_2)\omega^2 - \Delta\rho g k + (\omega^2 + gk)\Delta\rho \mathrm{e}^{-2kh}\right) = 0 \quad (4.83)$$

erfüllt ist. Diese Gleichung besitzt die beiden Lösungen

$$\omega^2 = gk, \quad (4.84)$$
$$\omega^2 \approx gk\frac{\Delta\rho}{\rho_1 + \rho_2}\left(1 - \mathrm{e}^{-2kh}\right) \quad (4.85)$$
$$\approx \frac{1}{2}N^2 kh \left(1 - \mathrm{e}^{-2kh}\right). \quad (4.86)$$

Bei der zweiten Lösung wurde die Dichtedifferenz gegenüber der Summe der Dichten vernachlässigt. Die erste Lösung ist die Frequenz von Schwerewellen für große Wassertiefen. Die zweite Lösung hat eine ähnliche Form, jedoch ist die Erdbeschleunigung g, die an der Oberfläche als rücktreibende Kraft wirkt, durch die um den Auftrieb reduzierte Erdbeschleunigung $g\frac{\Delta\rho}{\rho_1+\rho_2}$, die an der Grenzschicht als rücktreibende Kraft wirkt, ersetzt worden. Die Frequenz dieser Wellen ist bei gleicher Wellenzahl weitaus geringer als die der Oberflächenschwerewellen. Entsprechend geringer sind auch die Phasen- und die Gruppengeschwindigkeit. Die Dispersionseigenschaften sind ansonsten äquivalent zu denen der Schwerewellen auf einem ungeschichteten Gewässer. Die Größe

$$N^2 = g\frac{\Delta\rho}{h\rho} \tag{4.86}$$

ist uns in allgemeinerer Form bereits bei der Diskussion der statischen Stabilität einer Wassersäule begegnet und hieß dort (noch etwas unmotiviert) Bouyancy - Frequenz.

Bisher haben wir nur gezeigt, daß ein geschichtetes Gewässer gegenüber einem ungeschichteten weitere Freiheitsgrade aufweist. Man kann erwarten, daß für eine kompliziertere oder gar kontinuierliche Schichtung eine große Mannigfaltigkeit von Frequenzen auftritt, deren Betrag jedoch noch kleiner als die eben diskutierte Lösung sein würde, da die entsprechenden rücktreibenden Kräfte kleiner wären. Wie die zugehörigen hydrodynamischen Felder aussehen, hängt von den antreibenden Kräften ab. Dieses Problem soll hier nicht diskutiert werden. Insbesondere sollten die Moden durch die (hier vernachlässigten) Advektionsterme gekoppelt sein, so daß die einzelnen Moden eine Dämpfung bzw. Frequenzverschiebung erfahren würden. (Siehe z.B. gekoppelte Oszillatoren). Andere wichtige Probleme, die hier nicht betrachtet werden können, sind z.B. die Energetik interner Wellen, die Stokes-Drift in geschichteten Gewässeren, Vermischungsprozesse durch Brechung interner Wellenmoden, Wellenbeugung und Reflexion in horizontal variablen Schichtungen u.a..

4.3. The shallow water equations

In the previous sections an idealised problem of linear wave dynamics was solved analytically. Even for this simple problem approximations and simplifications are needed. We expect more difficulties for wind forced motion, stratified motion or interaction of waves with irregular bottom topography. A general finding in experiments is that for realistic situations the complete spectrum of waves is excited, but the major part of the energy is confined to a small frequency band. It may be of some help, to modify the general wave equations in such a way that the solution covers only the desired phenomena. This reduces the complexity of the problem considerably.

Here we investigate the "shallow water equations" valid for oceans of limited depth and for waves with wave length larger than the ocean depth.

The homogeneous case

We return to surface gravity waves for an unstratified ocean. The dispersion relation was

$$\omega^2 = gk\tanh(kH). \tag{4.87}$$

For waves with wave length exceeding the water depth we have found the approximation

$$\omega^2 \approx gHk^2. \tag{4.88}$$

We have two important properties of these waves:

- waves or non-dispersive,
- their potential is independent of depth. Hence, also the pressure perturbation p_e is depth independent.

From the surface boundary condition it follows

$$p_e = \rho g \xi, \tag{4.89}$$

which holds now for the total water column.

This allows us to write the Eulerian equations in terms of the sea surface elevation instead of the pressure. (Note, the gradients mof the hydrostatic pressure vanish, since the density is horizontally uniform.)

$$\frac{\partial u}{\partial t} = -g\frac{\partial \xi}{\partial x}, \tag{4.90}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \xi}{\partial y}. \tag{4.91}$$

This reduces the number of dynamic quantities. Comparing with the previous section we see, that the velocites are depth independent up to the order z^2. This does not apply to the vertical velocity, since the bottom and surface boundary conditions must be considered. However, the vertical velocity can be eliminated completely using the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{4.92}$$

u and v are independent of z and the vertical integral can be carried out.

$$w(\xi) - w(-H) = -\int_{-H}^{\xi} dz \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right). \tag{4.93}$$

The boundary conditions are found by applying the operator

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla} \tag{4.94}$$

to the equations defining surface and bottom,

$$z - \xi(xyt) = 0, \tag{4.95}$$

$$z + H(xy) = 0. \tag{4.96}$$

We get

$$w(0) - \frac{\partial \xi}{\partial t} - u\frac{\partial \xi}{\partial x} - v\frac{\partial \xi}{\partial y} = 0, \quad (4.97)$$
$$w(-H) + u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} = 0. \quad (4.98)$$

Now we exchange the horizontal derivatives and the vertical integral in the continuity equations. We account for the horizontal variability of the sea floor and the sea surface by the appropriate Leibniz rules. These cancel exactly the non-linear terms in the boundary conditions for w and we find

$$\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \quad (4.99)$$

U and V are the vertically integrated velocity components. Writing also the Euler equations in terms of these variables we find

$$\frac{\partial U}{\partial t} = -c^2\frac{\partial \xi}{\partial x}, \quad (4.100)$$
$$\frac{\partial V}{\partial t} = -c^2\frac{\partial \xi}{\partial y}. \quad (4.101)$$

($c^2 = gH$ is the phase velocity of gravity waves in shallow water approximation) After cross differencing we get an equation for the sea level alone,

$$\frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial x}c^2\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial y}c^2\frac{\partial \xi}{\partial y}, \quad (4.102)$$

or for a flat bottom ($H = const$)

$$\frac{\partial^2 \xi}{\partial t^2} - c^2\Delta_H \xi = 0. \quad (4.102)$$

This equation is hyperbolic and is called a *wave equation.* Inserting the shallow water approximation for the horizontal velocity from the previous chapter shows, that it is solution.

What about the vertical velocity? From $p_e = \rho g \xi$ is follows

$$\frac{\partial w}{\partial t} = 0. \quad (4.103)$$

This would prevent the spreading of surface waves in clear contradiction to our wave equation. To find the vertical acceleration at least terms of the order z^2 for the pressure perturbation must be considered. In the sense of a decomposition with respect to the small parameter kH the vertical acceleration is of higher order.

As a result a shallow water system is *hydrostatic* to the order k^2z^2, the vertical acceleration does not contribute to the vertical pressure gradient. In turn we may promise to ask ever for solutions with $kH \ll 1$ only and can generally require the *hydrostatic approximation* to be valid. Hence, we have found a filter to restrict the solution to the long wave limit - the hydrostatic approximation. It is a fundamental approximation in oceanography, especially in numerical modelling.

The variables u, v and ξ are governed by equations of motion containing the time tendency of these variables. For this reason they are called *prognostic* variables. In hydrostatic approximation the vertical velocity can be diagnosed from the horizontal flow field using the equation of continuity, hence, this variable is called *diagnostic*. For the shallow water equations it depends linearly on depth. The time derivative of the diagnostic w is for sure different from the true vertical acceleration.

Das Zweischichtmodell

Wir kehren noch einmal zum Zweischichtenmodell zurück und betrachten dieses in hydrostatischer Näherung.

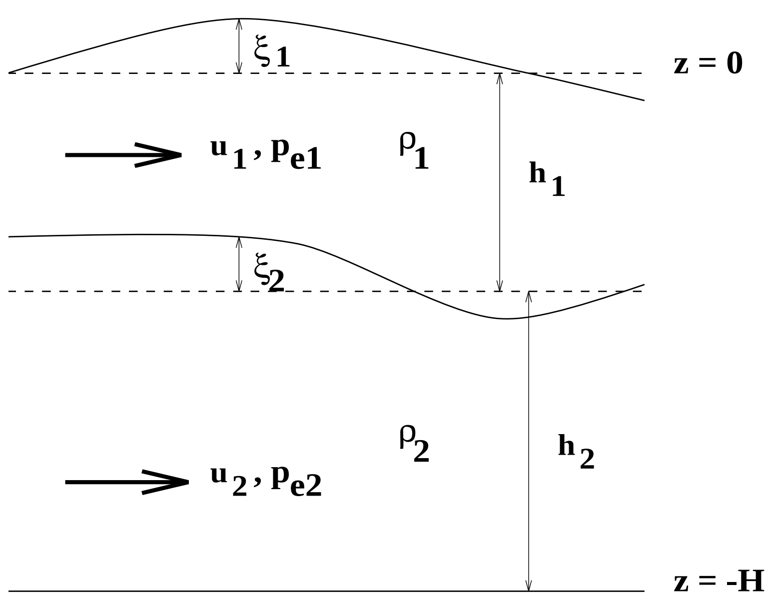


Figure 4.6. Geschichtetes Gewässer mit zwei überlagerten Wassermassen verschiedener Dichte.

Abbildung 4.6 zeigt den Aufbau der Schichtung. Der Exzeßdruck p_e ist von der Tiefe abhängig, jedoch innerhalb der Schichten konstant. Die Werte des Exzeßdruckes an den Schichtgrenzen, d.h. an der Oberfläche und an der inneren Grenze zwischen beiden

Schichten hatten wir bereits bestimmt. Dieser Werte gilt jetzt jeweils für die gesamte Schicht,

$$p_{e1} = \rho_1 g \xi_1, \tag{4.104}$$
$$p_{e2} = \rho_2 g \xi_2 + \rho_1 g (\xi_1 - \xi_2). \tag{4.105}$$

Die vertikalen Euler'schen Gleichungen lauten in beiden Schichten

$$\rho_1 \frac{\partial \mathbf{u}_1}{\partial t} = -\boldsymbol{\nabla}_h p_{e1} = -\rho_1 g \boldsymbol{\nabla}_h \xi_1, \tag{4.106}$$
$$\rho_2 \frac{\partial \mathbf{u}_2}{\partial t} = -\boldsymbol{\nabla}_h p_{e2} = -\rho_2 g \boldsymbol{\nabla}_h \xi_2 - \rho_1 g \boldsymbol{\nabla}_h (\xi_1 - \xi_2). \tag{4.107}$$

Der Index $_h$ steht für "horizontale Komponenten". Die Kontinuitätsgleichung kann über beide Schichten separat integriert werden. Dabei tritt die w-Komponente an der Obergrenze der unteren Schicht als Randbedingung der Integration in der oberen Schicht in Erscheinung. Wir nehmen wieder an, der Boden sei eben, d.h. $w(-H) = 0$.

$$\frac{\partial \xi_2}{\partial t} + h_2 \boldsymbol{\nabla}_h \cdot \mathbf{u}_2 = 0, \tag{4.108}$$
$$\frac{\partial (\xi_1 - \xi_2)}{\partial t} + h_1 \boldsymbol{\nabla}_h \cdot \mathbf{u}_1 = 0. \tag{4.109}$$

Wir können nun die Euler'sche Gleichung und die Kontinuitätsgleichung in der jeweiligen Schicht zu einer Gleichung kombinieren,

$$\frac{\partial^2 (\xi_1 - \xi_2)}{\partial t^2} = h_1 g \Delta_h \xi_1, \tag{4.110}$$
$$\frac{\partial^2 \xi_2}{\partial t^2} = h_2 \Delta_h (g \xi_1 - g' (\xi_1 - \xi_2)). \tag{4.111}$$

Die Auslenkungen ξ_1 und ξ_2 sind miteinander verkoppelt, d.h. die Beschleinigung von ξ_1 erfolgt nicht nur durch die Oberflächenauslenkung sondern zusätzlich durch eine Beschleunigung der ganzen oberen Schicht durch die darunterliegende. Andereseits erfolgt die Beschleunigung der Grenzschicht ξ_2 nicht nur durch die Gradienten der Oberflächenauslenkung sondern zusätzlich durch die Gradienten der Differenz $\xi_1 - \xi_2$. Die Schwerkraft wirkt hier jedoch durch den Auftrieb reduziert, daher tritt als Beschleunigung nur die reduzierte Erdbeschleunigung

$$g' = g \frac{\rho_2 - \rho_1}{\rho_2} \tag{4.111}$$

in Erscheinung. g' ist für ozeanische Bedingungen sehr klein, $\frac{g'}{g} \approx 0.003$.

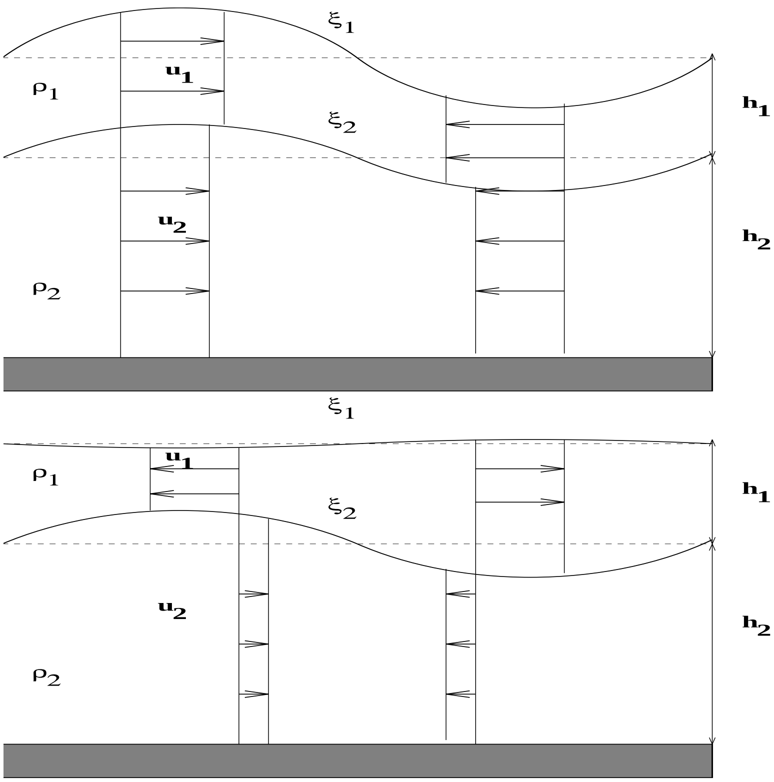


Figure 4.7. Der barotrope und der barokline Mode.

Beider Gleichungen können wir so linear kombinieren, daß nur noch ξ_1 und die Differenz $\xi_1 - \xi_2$ auftritt,

$$\frac{\partial^2 (\xi_1 - \xi_2)}{\partial t^2} = h_1 g \Delta_h \xi_1, \tag{4.112}$$

$$\frac{\partial^2 \xi_1}{\partial t^2} = Hg\Delta_h \xi_1 - h_2 g' \Delta_h (\xi_1 - \xi_2) . \tag{4.113}$$

Durch Einsetzen kann jetzt $\xi_1 - \xi_2$ eliminiert werden,

$$\frac{\partial^4 \xi_1}{\partial t^4} - Hg\Delta_h \frac{\partial^2 \xi_1}{\partial t^2} + h_2 h_1 g' \Delta_h \Delta_h \xi_1 = 0. \qquad (4.113)$$

Wir betrachten der Einfachkeit halber den eindimensionalen Fall. Ein allgemeiner Lösungsansatz ist wieder

$$\xi_1 = \mathrm{e}^{i\omega t - kx}. \qquad (4.113)$$

Damit ergibt sich eine Dispersionsreslation

$$\omega^4 - \omega^2 k^2 Hg + k^4 h_2 h_1 g' = 0 \qquad (4.113)$$

mit den zwei Lösungen

$$\omega_{bt,bk}^2 = \frac{gHk^2}{2}\left(1 \pm \sqrt{1 - 4\frac{h_1 h_2}{H^2}\frac{g'}{g}}\right). \tag{4.113}$$

Der zweite Term unter der Wurzel ist meistens viel kleiner als 1, so daß die Wurzel entwickelt werden kann. Damit lauten die beiden Lösungen

$$\omega_{bt}^2 \approx gHk^2, \tag{4.114}$$

$$\omega_{bk}^2 \approx g'\frac{h_2 h_1}{H}k^2. \tag{4.115}$$

Die Phasen- und die Gruppengeschwindigkeiten sind

$$c_{bt} = \sqrt{gH}, \tag{4.116}$$

$$c_{bk} = \sqrt{g'\frac{h_2 h_1}{H}}. \tag{4.117}$$

Wie bereits für den Fall tiefen Wassers abgeleitet, gibt es zwei Lösungen mit verschiedener Dispersionsrelation. Die erste entspricht näherungsweise der Dispersionsrelation in einem homogenen Flachwassersystem.

Um zu zeigen, daß nicht nur die Dispersionsrelation übereinstimmt, betrachten wir die Beziehung zwischen ξ_2 und ξ_1, die aus den Differentialgleichungen folgt. Mit der Fourierdarstellung findet man sofort

$$\xi_2 = \xi_1\frac{\omega^2 - h_1 g k^2}{\omega^2}, \tag{4.118}$$

$$u_2 = u_1\left(1 - g'\frac{h_1 k^2}{\omega^2}\right). \tag{4.119}$$

Für $\omega = \omega_{bt}$ erhält man

$$\xi_2 = \xi_1\left(1 - \frac{h_1}{H}\right), \tag{4.120}$$

$$u_2 = u_1\left(1 - \frac{g'}{g}\frac{h_1}{H}\right) \approx u_1. \tag{4.121}$$

Wir erinnern uns an die Diskussion des homogenen Flachwassermodells. Die Vertikalgeschwindigkeit als diagnostische Variable hängt in diesem Fall linear von der Tiefe ab. Dasselbe gilt dann auch für das Zeitintegral von w, d.h. die Auslenkung einer Schicht in einer bestimmten Tiefe. Für eine Schicht in der Tiefe $-h_1$ würde aber gerade obiges Ergebnis folgen, d.h. der zu ω_{bt} gehörende Mode ist durch ein Stromfeld gekennzeichnet, das näherungsweise wie das eines ein homogenen Modells aufgebaut ist. Die Schichtung wirkt sich auf Wellen dieses Modes nicht aus. Da beide Grenzflächen parallel ausgelenkt werden, sind Flächen gleicher Dichte näherungsweise auch Flächen gleichen Drucks. Dieser Mode wird daher als *barotroper* Mode bezeichnet.

Im zweiten Mode sind Oberfläche und innere Grenzschicht engegengesetzt ausgelenkt,

$$\xi_2 = -\xi_1 \frac{Hg - h_2 g'}{h_2 g'} \approx -\xi_1 \frac{Hg}{h_2 g'}, \tag{4.122}$$

$$u_2 = -u_1 \frac{h_1}{h_2}. \tag{4.123}$$

Da Dichteflächen und Druckflächen nicht zusammenfallen, spricht man vom *baroklinen* Mode. Eine kleine Oberflächenauslenkung entspricht einer sehr großen Auslenkung der inneren Grenzschicht. Die Geschwindigkeiten in beiden Schichten sind von derselben Größenordnung, jedoch entgegengesetzt gerichtet. Da die Dichteunterschiede im Ozean allgemein sehr klein sind, ist der Unterschied in der Ausbreitungsgeschwindigkeit gewaltig.

4.4. Waves establishing flow fields

At this point the fundamental role of waves for signal propagation in the ocean can be highlighted first time. A colloquial approach considers ocean current as fundamental for the spreading of matter and energy. This is not wrong, but does not tell anything on how these flow fields are established. Surely, the local flow is not only the results of local forces but depends also on remote forces. As an example we consider a simplified model for the exchange of water between two basins through narrow straits. For example between North Sea and Baltic Sea, the Black Sea and the Mediterranian or the Mediterranian and the Atlantic. We do not care for a moment on the Coriolis force. In this case the flow is approximately one-dimensional.

Assuming an initial sea surface elevation $\xi = \xi_0(x) = \eta\theta(x)$ at $t = 0$, but zero velocity. The solution of the wave equation is simply,

$$\xi(xt) = \frac{1}{2} \left(\xi_0(x + ct) + \xi_0(x - ct) \right). \tag{4.124}$$

Velocity follows from the Euler equation

$$U(xt) = -\frac{1}{2} \frac{c}{H} \left(\xi_0(x + ct) - \xi_0(x - ct) \right). \tag{4.125}$$

Figure 4.8 shows the spreading of waves from an initial step-like sea level elevation.

Remarkably, the flow is established only by waves. The information on the existing perturbation of the sea level spreads with phase velocity in form of a wave front. Since we use the shallow water approximation the waves are non-dispersive and the shape of the front is conserved. The group velocity determines the established flow, which remains finite although the sea level is elevated step-like. We have a wave controlled ocean dynamics.

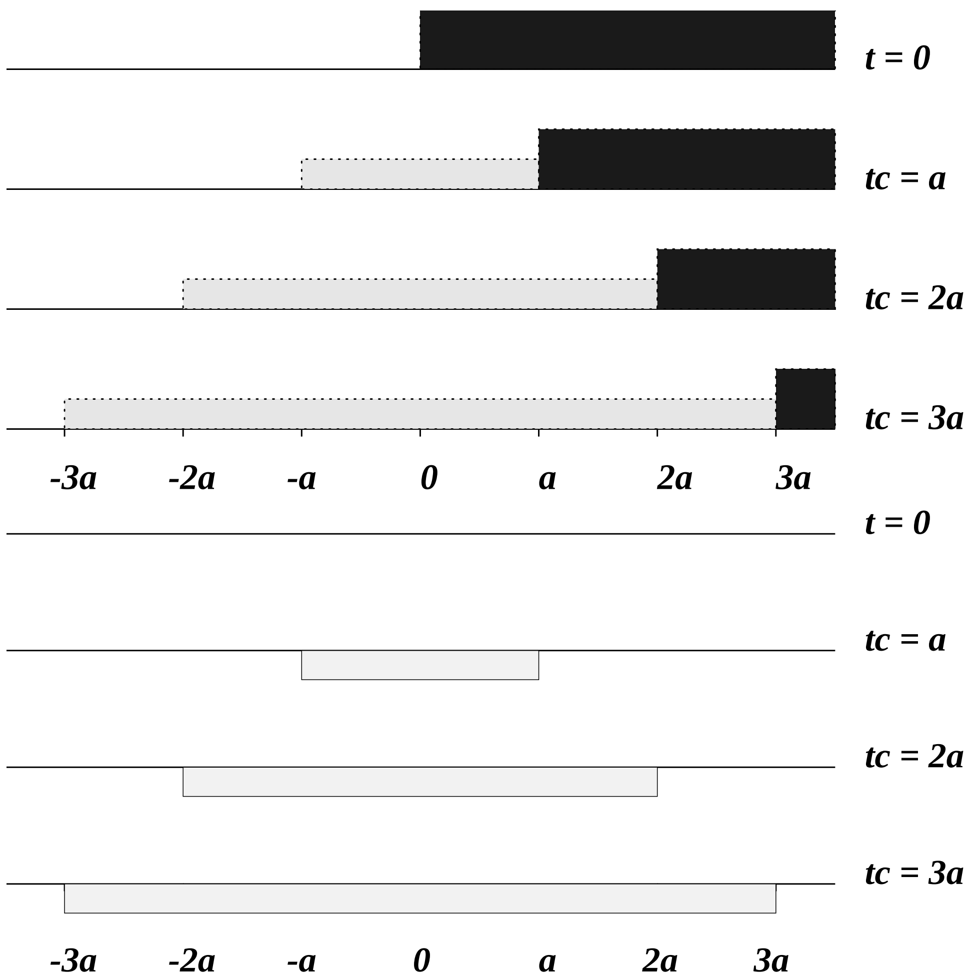


Figure 4.8. The spreading of waves in resonse to an initial step-like elevation, upper: sea level, lower: velocity

Chapter 5

Shallow water waves on a rotating planet

We start now the discussion of wave, establishing flow field, but on a rotating planet. This chapter includes many other aspects of flow fields like geostrophy and vorticity conservation.

5.1. The shallow water equations for the f-plane

We derive the shallow water equations and introduce potential vorticity as a conserved flow variable. With the Rossby adjustment problem we show, that potential vorticity conservation provides the missing information to complete the geostrophic equations.

Originally the shallow water equations have been derived from a general wave-like solution of the ocean equations. As a result we have found the hydrostatic approximation as valid in the long wave and shallow water approximation. Now we introduce the hydrostatic approximation as general method to simplify our equations.

As a first step we introduce the Coriolis force in f-plane approximation, the vertical component of the Coriolis force is small compared with gravitation. The Euler equations, now including the nonlinear terms, are

$$\frac{Du}{Dt} - fv = -g\frac{\partial \xi}{\partial x}, \tag{5.1}$$

$$\frac{Dv}{Dt} + fv = -g\frac{\partial \xi}{\partial y}. \tag{5.2}$$

the equation of continuity was

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.3}$$

The pressure was

$$p(z) = \rho g \xi - \rho g z + p_a, \tag{5.4}$$

which explaines the simplified pressure terms in the Euler equations. Again, u and v are independent of depth (for this reason the vertical derivative in the operator D/Dt does not contribute.). The equation of continuity can be integrated vertically,

$$w(\xi) - w(-H) = -\int_{-H}^{\xi} dz \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \tag{5.5}$$

With the surface and bottom boundary conditions,

$$\frac{D}{Dt}(z - \xi(x,y)) = 0, \tag{5.6}$$

$$\frac{D}{Dt}(z + H(x,y)) = 0, \tag{5.7}$$

or equivalently,

$$w(\xi) - w(-H) = \frac{D(\xi(x,y) + H)}{Dt} \tag{5.8}$$

the continuity equation becomes

$$\frac{1}{\xi + H}\frac{D(\xi + H)}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.9}$$

With the Leibniz - rules the equivalent for for the vertically integrated velocities can be derived,

$$\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \tag{5.10}$$

with

$$U = \int_{-H}^{\xi} dz\, u, \qquad V = \int_{-H}^{\xi} dz\, v. \tag{5.11}$$

Hence our set of basic equations in hydrostatic and shallow water approximation reads now

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \xi}{\partial x}, \tag{5.12}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \xi}{\partial y}, \tag{5.13}$$

$$\frac{1}{\xi + H}\left(\frac{\partial \xi}{\partial t} + u\frac{\partial(\xi + H)}{\partial x} + v\frac{\partial(\xi + H)}{\partial y} \right) = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \tag{5.14}$$

We express these equations completely in terms of the total depth, D and the vertically integrated horizontal velocities, $\mathbf{U}$,

$$U = Du, \qquad V = Dv, \qquad D = \xi + H. \tag{5.15}$$

With

$$\frac{\partial U}{\partial t} = D\frac{\partial u}{\partial t} + u\frac{\partial D}{\partial t}, \tag{5.16}$$

surface and bottom boundary conditions and Leibniz rules, the set of equations reads

$$\frac{\partial U}{\partial t} + \frac{\partial \frac{U^2}{D}}{\partial x} + \frac{\partial \frac{UV}{D}}{\partial y} - fV = -gD\frac{\partial \xi}{\partial x}, \tag{5.17}$$

$$\frac{\partial V}{\partial t} + \frac{\partial \frac{UV}{D}}{\partial x} + \frac{\partial \frac{V^2}{D}}{\partial y} + fU = -gD\frac{\partial \xi}{\partial y}, \tag{5.18}$$

$$\frac{\partial \xi}{\partial t} = -\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right). \tag{5.19}$$

For a flat bottom, $H = const$, small velocities and small sea level elevation the linear approximation follows,

$$\frac{\partial U}{\partial t} - fV + gH\frac{\partial \xi}{\partial x} = 0, \tag{5.20}$$

$$\frac{\partial V}{\partial t} + fU + gH\frac{\partial \xi}{\partial y} = 0, \tag{5.21}$$

$$\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0. \tag{5.22}$$

or equivalently

$$\frac{\partial u}{\partial t} - fv + g\frac{\partial \xi}{\partial x} = 0, \tag{5.23}$$

$$\frac{\partial v}{\partial t} + fu + g\frac{\partial \xi}{\partial y} = 0, \tag{5.24}$$

$$\frac{1}{H}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.25}$$

5.1.1. The steady case - geostrophic flow

We may ask for a steady state solution of the shallow water equations. As a result we find three equations for the three quantities u, v and ξ,

$$-fv + g\frac{\partial \xi}{\partial x} = 0, \tag{5.26}$$

$$fu + g\frac{\partial \xi}{\partial y} = 0, \tag{5.27}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{5.28}$$

The first two equations describe a so called geostrophic balance. The pressure gradient is balanced by the Coriolis force and the flow follows isobars instead of equilibrating pressure gradients. The third equation defines the diagnosed vertical derivative of the vertical velocity, which is for a geostrophically balanced flow zero.

Moreover, the third equation can be derived from the horizontal equations by cross-differencing. Hence, the equations are not linearly independent and a unique solution does not exists.

This is an important finding - the steady state shallow water equations cannot uniquely solved. The geostrophic balance is not a prognostic relation which defines neither the pressure nor the horizontal velocities. To find the geostrophic steady state more information on the history of flow field is needed. In the next sections we will show, how a geostrophically balanced flow may develop. In more detail, we will highlight the role of waves in the establishment of geostrophically balanced flows.

5.1.2. More conservation laws - potential vorticity

Before we consider an example for a so called geostrophic adjustment process we consider another budget equation. To solve the shallow water equations, the velocity could be eliminated by cross differencing and a single equation for the pressure (sea level) can be derived. This will be done later. First we consider a different approach developped by Lagrange. With the identity

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\partial (u^2+v^2)}{\partial x} + v\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right), \tag{5.29}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{2}\frac{\partial (u^2+v^2)}{\partial y} + u\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \tag{5.30}$$

the momentum equations read

$$\frac{\partial u}{\partial t} - v(f+\chi) + \frac{\partial B}{\partial x} = 0, \tag{5.31}$$

$$\frac{\partial v}{\partial t} + u(f+\chi) + \frac{\partial B}{\partial y} = 0. \tag{5.32}$$

The Bernoulli function, B, is

$$B = g\xi + \frac{1}{2}\left(u^2+v^2\right). \tag{5.33}$$

It stands for the mechanical energy of a fluid element. Rotating movement in a fluid is related to the curl of the horizontal currents,

$$\chi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{5.34}$$

The specific name of χ is *relative vorticity*. Together with the *planetary vorticity*, f, it forms the *absolute vorticity*, $f+\chi$. This quantity corresponds to the total angular momentum of a fluid element in the inertial system. The relative vorticity is twice of the relative angular velocity.

Applying the curl-operator eliminates the Bernoulli function,

$$\frac{D_h\chi}{Dt} + (f+\chi)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0, \tag{5.35}$$

or equivalently,

$$\frac{1}{f+\chi}\frac{D_h\chi}{Dt} + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{5.36}$$

The operator $D_h\ Dt$ is the horizontal part of the Lagrangian time derivative,

$$\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}. \tag{5.37}$$

This is an important finding. Horizontal flow divergency corresponds to changes of the absolute vorticity. If f is approximately constant, flow divergence causes local rotation.

With help of the continuity equation the flow divergence can be expressed in terms of sea level changes,

$$\frac{D_h\Pi}{Dt} = 0, \tag{5.38}$$

$$\Pi = \frac{f+\chi}{H+\xi}. \tag{5.39}$$

Π is called *potential vorticity* (Rossby, 1940). This quantity is conserved for long-wave (shallow water) frictionless motion of fluids in hydrostatic balance.

5.1.3. The linearised shallow water equations

In many cases the non-linear terms (advection of momentum) are small and a linear approximation applies. In terms of scales for time, T, horizontal length scale, L and velocity scale, U, (do not mix with the vertically integrated velocity), the local time derivative, the advection terms and the Coriolis term read

$$\frac{U}{T} + \frac{U^2}{L} + fU = u\left(\frac{1}{T} + \frac{U}{L} + f\right). \tag{5.40}$$

The magnitude of the time derivative is determined by the other terms. Hence, if the *Rossby number*

$$Ro = \frac{U}{Lf} \ll 1, \tag{5.41}$$

the advective terms can be neglected. The approximate set of equations

$$\frac{\partial U}{\partial t} - fV = -gD\frac{\partial \xi}{\partial x}, \tag{5.42}$$

$$\frac{\partial V}{\partial t} + fU = -gD\frac{\partial \xi}{\partial y}, \tag{5.43}$$

$$\frac{\partial \xi}{\partial t} = -\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right), \tag{5.44}$$

is still non-linear, since the depth D depends on the sea level height.

To find a solution, the velocities can be eliminated. Deriving the first equation to x and the second to y we find

$$\frac{\partial}{\partial t}\left(U_x + V_y\right) - f\left(V_x - U_y\right) = -g\boldsymbol{\nabla}_h \cdot \left(D\boldsymbol{\nabla}_h \xi\right). \tag{5.45}$$

The time derivative of the flow divergence can be eliminated with the continuity equation. To eliminate the vorticity, we use the cross differentiation

$$\frac{\partial}{\partial t}\left(U_y - V_x\right) - f\left(U_x + V_y\right) = -gJ(D,\xi), \tag{5.46}$$

with the Jacobian

$$J(D,\xi) = D_x\xi_y - D_y\xi_x. \tag{5.47}$$

The final equation in term of the sea level elevation is

$$\frac{\partial}{\partial t}\left(\xi_{tt} + f^2\xi - g\boldsymbol{\nabla}_h \cdot (D\nabla_h\xi)\right) = gfJ(D,\xi). \tag{5.48}$$

Steady flow

For constant flow, the velocity and pressure are in geostrophic balance,

$$-fv = -g\frac{\partial\xi}{\partial x}, \tag{5.49}$$

$$+fu = -g\frac{\partial\xi}{\partial y}. \tag{5.50}$$

This looks similar to the velocity potential derived for surface gravity waves. However, this relation is completely different and depends on the existence of rotation. In the limit $f \to 0$ it does not converge towards the non-rotating potential representation of the flow. The geostrophic balance implies

$$u\xi_x + v\xi_y = \mathbf{u_h} \cdot \boldsymbol{\nabla}_\mathbf{h}\xi = \mathbf{0}, \tag{5.51}$$

the flow follows lines of constant sea level elevation (isobars). But also the Jacobian $J(D,\xi)$ vanishes for a steady state,

$$D_x\xi_y - D_y\xi_x = 0. \tag{5.52}$$

With the geostrophic balance we get also

$$D_x\xi_y - D_y\xi_x = -\frac{f}{g}\left(D_x u - D_y v\right) = -\frac{f}{g}\mathbf{u_h} \cdot \boldsymbol{\nabla}_\mathbf{h}\mathbf{D} = 0. \tag{5.53}$$

So the flow follows also lines of constant depth. Since usually the sea level elevation is much smaller than the depth, it can be said, that the flow follows the isobath. Additionally, the sea level elevation must reflect the pattern of the sea floor, to fulfill both the conditions simultaneously, follow isobaths and isobars.

This finding is not only a marginal consequence of the shallow water equations, but a very general property of fluids at rotating planets. Indeed, the *Taylor-Proudman theorem* states, that a geostrophic flow within a weekly stratified rotating fluid is always independent of depth. This implies that, since the flow at the bottom follows the isobaths, it does this also at the surface. Hence, in a weakly stratified ocean, the sea floor influences also the surface currents. Together with the conservation of potential vorticity an unstratified ocean can be considered as consisting of vertical *vortex columns* or *vortex tubes*. These vortex columns are moving such a way the potential vorticity remains unchanged. If they would be crossing isobaths, their length would be changed corresponding to a change of their diameter. This requires a local vortex flow, which in turn requires energy. Hence, flow across isobaths generates flow with horizontal scales of the gradients of the sea floor. This is in geostrophic balance itself after some time - as shown above it follows the isobaths.

Wave like solution

To study the wave like solutions we consider free waves. The equation 5.48 for the sea surface elevation is still non-linear. As a first step we assume a flat bottom and negelect all terms quadratic in the sea level elevation,

$$\frac{\partial}{\partial t}\left(\xi_{tt} + f^2\xi - gH\Delta\xi\right) = 0. \tag{5.54}$$

With the exponential ansatz

$$\xi = \xi_0 e^{i\omega t - k_x x - k_y y} \tag{5.55}$$

we find a condition that a solution exists,

$$i\omega(-\omega^2 + f^2 + gHk^2) = 0. \tag{5.56}$$

Hence, either the frequency must vanish (the steady geostrophic solution) or the frequency and wave number fulfill the dispersion relation

$$\omega^2 = f^2 + gHk^2 \tag{5.57}$$

We recognise the phase velocity for shallow water waves in the non-rotating case,

$$c^2 = gH. \tag{5.58}$$

For the waves at the rotating planet we find,

$$\begin{aligned} c_p &= \frac{\omega}{k} = \sqrt{c^2 + \frac{f^2}{k^2}} = c\sqrt{1 + \frac{1}{k^2 R_0^2}} \\ , c_g &= \frac{c}{\sqrt{1 + \frac{1}{k^2 R_0^2}}} = c\frac{\sqrt{\omega^2 - f^2}}{\omega}. \end{aligned} \tag{5.58}$$

Waves with this dispersion relation are often called Poincaré-waves. The phase velocity exceeeds the group velocity, Poincaré-waves show normal dispersion. The quantity

$$R_0 \;=\; \frac{\sqrt{gH}}{f}, \tag{5.59}$$

is called *Rossby radius.* It depends on the phase speed of gravity waves in the non-rotating limit and the Coriolis parameter, f. It is a typical horizontal length scale for oceanic processes. If the wave length $\lambda = 2\pi/k$ of the waves are much smaller than the Rossby radius, the effect of rotation becomes small. These processes can be described by the simpler equations for the non-rotating case. Otherwise the Coriolis terms must be taken into account.

5.1.4. An application - the Rossby adjustment problem

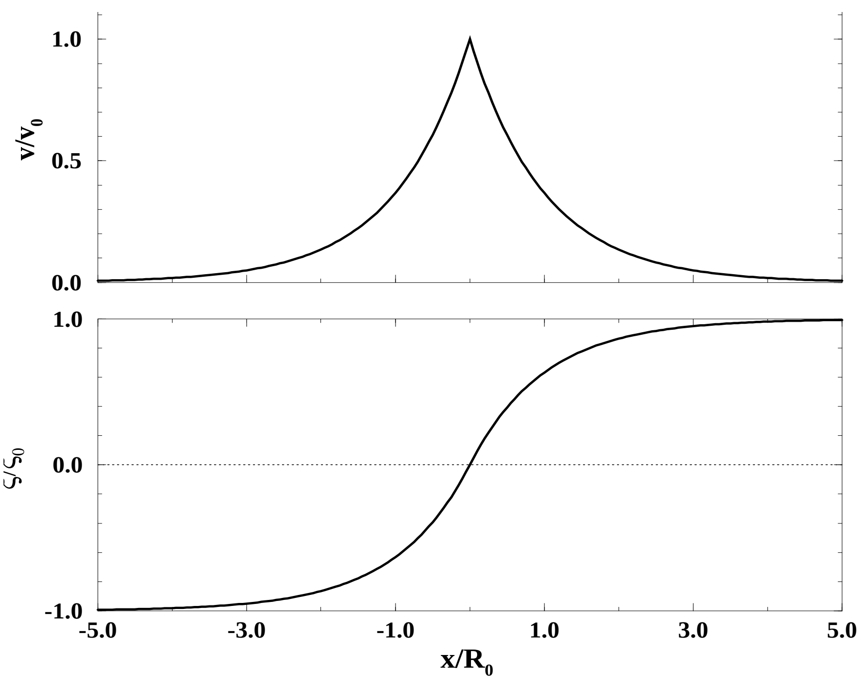


Figure 5.1. Geostrophic currents and sea level elevation as final state developing from a initial step like perturbation.

Now we consider a first application. Assume an initial step-like sea surface elevation at

$x = 0$,

$$\begin{aligned} \xi(t=0) &= \xi_0 \mathrm{sgn}(x), \\ u(t=0) &= 0, \quad v(t=0) = 0. \end{aligned} \tag{5.59}$$

We do not solve equation 5.54 directly. The reason is, that we need three initial conditions for the sea level elevation. To find them, we discuss the solution in terms of conservation of potential vorticity. For simplicity we assume $H = const$. From the velocity equations we find

$$\frac{\partial^2 \xi}{\partial t^2} - c^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) = -fH\chi. \tag{5.60}$$

We do not eliminate the relative vorticity as previously but use our finding on the conservation of potential vorticity,

$$\Pi = \frac{f+\chi}{H+\xi} \approx \frac{\chi}{H} - f\frac{\xi}{H^2} = const. \tag{5.61}$$

From the initial conditions we can determine the potential vorticity,

$$\Pi = -f\frac{\xi_0}{H^2}. \tag{5.62}$$

Hence, the inhomogeneity in the sea level equation is know in terms of the sea level,

$$fH\chi = f^2 H\Pi + f^2\xi. \tag{5.63}$$

We have to solve the equation inhomogenious equation

$$\frac{\partial^2 \xi}{\partial t^2} - c^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + f^2\xi = -fH^2\Pi. \tag{5.64}$$

We could define a Green's function and solve this equation for any potential vorticity Π. But as a first step we discuss the solution of the stead state equation

$$-c^2 \left(\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} \right) + f^2\xi = f^2\xi_0 \mathrm{sgn}(x). \tag{5.65}$$

We can guess the solution

$$\xi = \xi_0 \mathrm{sgn}(x) \left(1 - \mathrm{e}^{-\frac{|x|}{R_0}}\right), \tag{5.66}$$

$$u = 0, \tag{5.67}$$

$$v = c\frac{\xi_0}{H} \mathrm{e}^{-\frac{|x|}{R_0}}. \tag{5.68}$$

Compared with the non-rotating case this is a new finding. The initial sea surface elevation does not disappear but remains conserved, especially far away from the initial disturbance. The velocity field that has emerged, follows the isobars. It is in geostrophic balance with the pressure gradients. Remarkably, we have now a unique solution where sea level elevation and velocity are in geostrophic balance. Here, conservation of potential vorticity adds the missing information needed to overcome the ambiguity of the geostrophic equations. The potential vorticity keeps information on the initial state, which makes the final steady state unique.

The response to the sea level disturbance is confined to an area defined by the Rossby radius R_0. It serves as a typical horizontal scale for the developing flow field.

Now we consider the changes of the potential and kinetic energy during the adjustment process. The potetial energy released is

$$\begin{aligned} \Delta E_{pot} &= 2\frac{1}{2}\rho g \xi_0^2 \int_0^\infty dx \left(\left(1 - \mathrm{e}^{-\frac{x}{R_0}}\right)^2 - 1 \right), && (5.69) \\ &= -\frac{3}{2}\rho g \xi_0^2 R_0, && (5.70) \end{aligned}$$

the kinetic energy gained by the velocity field is

$$\begin{aligned} \Delta E_{kin} &= 2\frac{1}{2}\rho g \xi_0^2 \int_0^\infty dx \mathrm{e}^{-2\frac{x}{R_0}}, \\ &= \frac{1}{2}\rho g \xi_0^2 R_0. && (5.70) \end{aligned}$$

Obviously the major part of the released potential energy is not found in the kinetic energy of the currents. From our previous knowledge on adjustment processes in the non-rotating case we can assume, that this energy is carried away by waves. This will be considered below.

We define the Fourier transformation of ξ with respect to x and t. The initial step like eleveation is independent of y and this special symmetry is never disturbed.

$$\xi(k\omega) = \int_{-\infty}^{+\infty} dx \int_0^{+\infty} dt \xi(xt) \mathrm{e}^{i(\omega t - kx)}. \qquad (5.70)$$

The time integral starts at $t = 0$, since the initial disturbance is released at $t = 0$. To define the time integral for large times we introduce a small imaginary part of ω, $\omega \to \omega + i\varepsilon$. This ensures convergence of the time integral. However, we make ε explicit only, if it is needed.

We transform all terms of the differential equation. For the term with the second derivative with respect to the time partial integration is necessary,

$$\begin{aligned} \int_0^{+\infty} dt \frac{\partial^2 \xi}{\partial t^2} \mathrm{e}^{i\omega t} &= -\left.\frac{\partial \xi}{\partial t}\right|_{t=0} - \int_0^{+\infty} dt \frac{\partial \xi}{\partial t} i\omega \mathrm{e}^{i\omega t} \\ &= i\omega \xi_0 - \omega^2 \xi(\omega). && (5.70) \end{aligned}$$

We have used the initial condition

$$\left.\frac{\partial \xi}{\partial t}\right|_{t=0} = 0, \tag{5.70}$$

which follows from the initial condition for the velocity and the continuity equation. For the transformation of the potential vorticity we use the relation

$$\int_0^{+\infty} dt e^{i\omega t} = \frac{i}{\omega + i\varepsilon}. \tag{5.70}$$

Hence, our differential equation reads now after transformation for t,

$$\left(f^2 - \omega^2\right) \xi - c^2 \frac{\partial^2 \xi}{\partial x^2} = \frac{i}{\omega} \left(f^2 - \omega^2\right) \xi_0. \tag{5.70}$$

To transform with repect to x we require, that ξ vanishes for large $|x|$,

$$\xi(x) \to \xi(x) e^{-\varepsilon |x|}. \tag{5.71}$$

Again, ε is is a small positive quantity.

Resolving our equation for ξ gives

$$\xi(k\omega) \;=\; \xi_0 \frac{f^2 - \omega^2}{i\omega \left(\omega^2 - f^2 - c^2 k^2\right)}. \tag{5.72}$$

From the denominator we recognise our dispersion relation, which determines the spectrum of the solution. $\xi(k\omega)$ is an analytical function in the complex plane except for single poles at

$$\omega \;=\; -i\varepsilon, \tag{5.73}$$

$$\omega \;=\; \pm\omega_k = \pm\sqrt{f^2 + c^2 k^2} - i\varepsilon. \tag{5.74}$$

Notably, all poles are near the real axis, but slightly shifted into the lower half plane. We can now use Cauchy's integral theorem to carry out the ω-integral,

$$\oint \frac{d\omega}{2\pi i} \frac{f(\omega)}{\omega - \omega_0} \;=\; f(\omega_0). \tag{5.75}$$

The integration path must be chosen anti-clockwise and must enclose ω_0. Otherwise it can be arbitrarily deformed.

The inverse Fourier transforation reads

$$\xi(kt) \;=\; \xi_0(k) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{e^{-i\omega t}}{\omega \left(\omega^2 - f^2 - c^2 k^2\right)} \tag{5.76}$$

The integration path along the real ω-axis can be completed by a semi-circle in the upper- or lower half plane. In the upper half plane, the imaginary part of ω is positive, and $\mathrm{Im}(\omega) t$

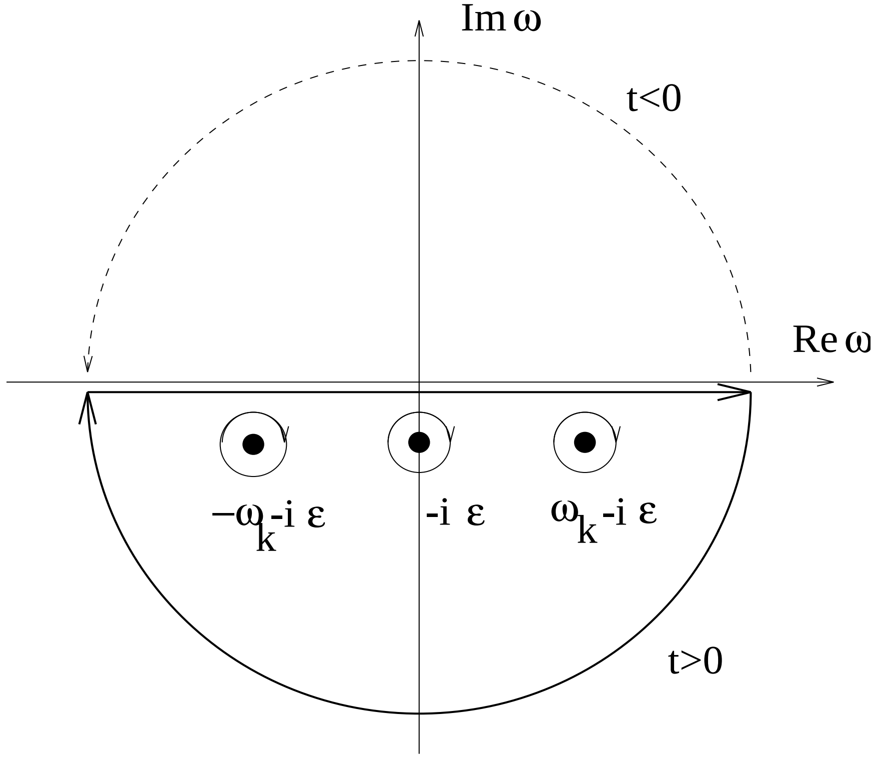


Figure 5.2. Integration path in the complex ω-plane. For $t < 0$ there are no poles enclosed in the upper half-plane, for $t > 0$ all poles in the lower half-plane are enclosed by the integration path.

diverges for $t > 0$ and converges for $t < 0$. However, there are no singularities in the upper half plane. Consequently, the inverse Fourier transformation vanishes for $t < 0$. In the lower half plane the argument of the exponential function diverges for $t < 0$ and vanishes away from the real axis and $t > 0$. Hence, we close the integral in the lower half plane and find

$$\begin{aligned} \xi(kt) &= \xi_0(k) \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{\mathrm{e}^{-i\omega t}}{\omega\,(\omega^2 - f^2 - c^2k^2)} \\ &= -\theta(t)\xi_0(k) \oint \frac{d\omega}{2\pi i} \frac{\mathrm{e}^{-i\omega t}}{\omega\,(\omega^2 - f^2 - c^2k^2)}, \\ &= \xi_0(k) \left(1 + \frac{c^2k^2}{\omega_k^2} (\cos \omega_k t - 1) \right). \end{aligned} \tag{5.75}$$

The inverse transformation of ξ with repsect to k cannot be carried out analytically.

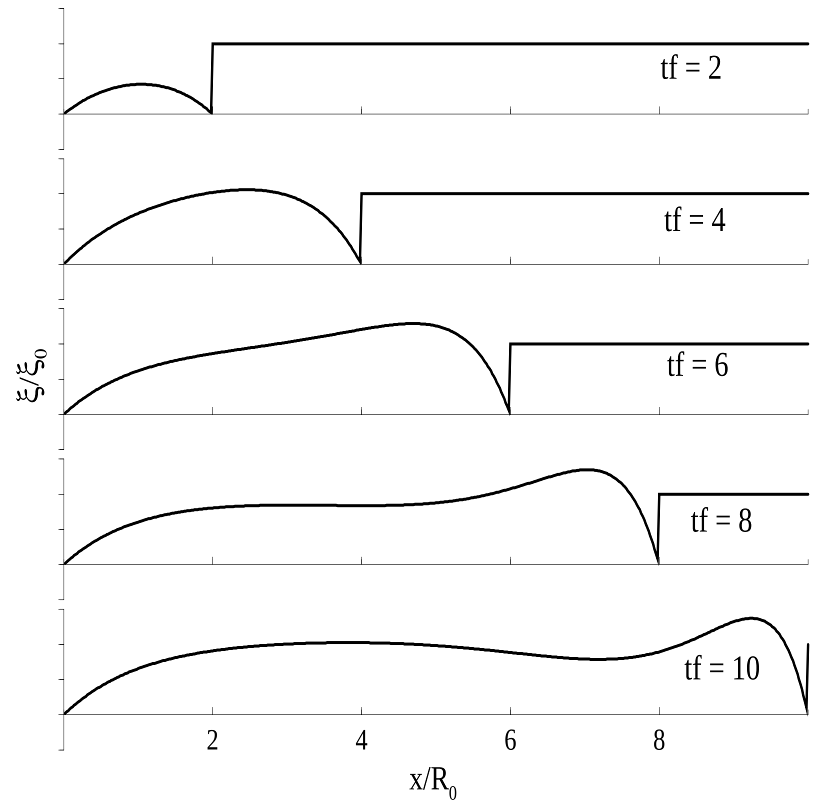


Figure 5.3. Sea level response to an initial step-like elevation at $x = 0$

However, with help of the continuity equation an expression for u can be derived,

$$\begin{aligned} u(kt) &= -\frac{1}{ikH}\frac{\partial \xi}{\partial t} \\ &= \frac{\xi_0(k)}{ikH}\frac{c^2k^2}{\omega_k}\sin\omega_k t. \end{aligned} \tag{5.75}$$

For a step like initial elevation we get with

$$\begin{aligned} \xi_0(k) &= \int_{-\infty}^{+\infty} dx \mathrm{e}^{ikx}\xi_0 \mathrm{sgn}(x) \\ &= -\xi_0\frac{2ik}{k^2+\varepsilon^2}, \end{aligned} \tag{5.75}$$

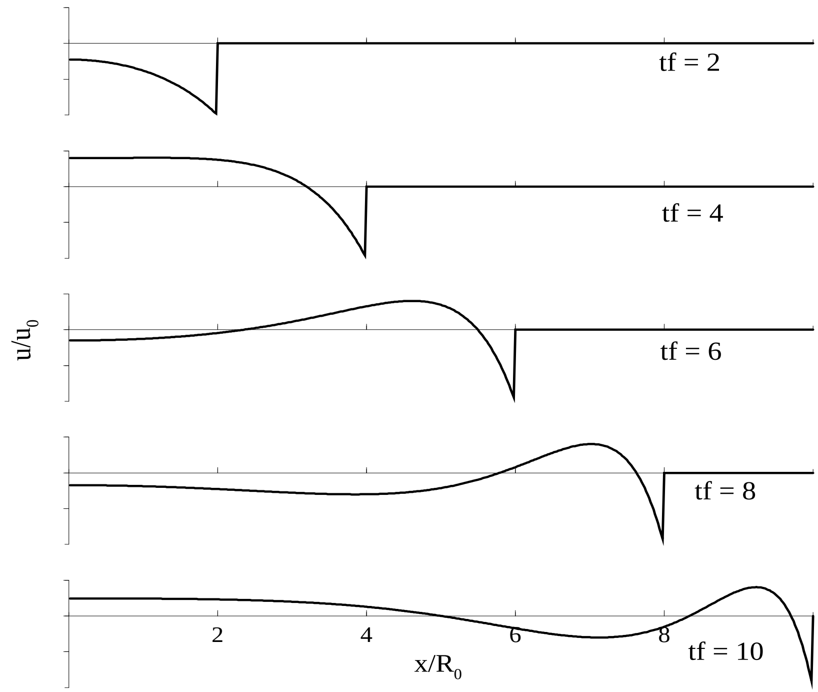


Figure 5.4. u-componente in response to an initial step-like elevation at $x = 0$

an analytical expression for u,

$$\begin{aligned} u(xt) &= -\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \frac{2\xi_0 \mathrm{e}^{ikx}}{H} \frac{c^2}{\omega_k} \sin \omega_k t \\ &= -c\frac{\xi_0}{H}\theta(ct - |x|) J_0 \left(f\sqrt{t^2 - \frac{x^2}{c^2}} \right) . \end{aligned} \tag{5.75}$$

The θ-functions arise from similar arguments like those used for the complex ω-integration.

With help of a little algebra using the continuity equation and the momentum equation,

$$\begin{aligned} H\frac{\partial u}{\partial x} &= \xi_0(x)\delta\left(t - \frac{|x|}{c}\right) \\ &\quad -\xi_0\theta(ct - |x|)\frac{fx}{\sqrt{t^2 - \frac{x^2}{c^2}}} J_1\left(f\sqrt{t^2 - \frac{x^2}{c^2}}\right) \end{aligned} \tag{5.75}$$

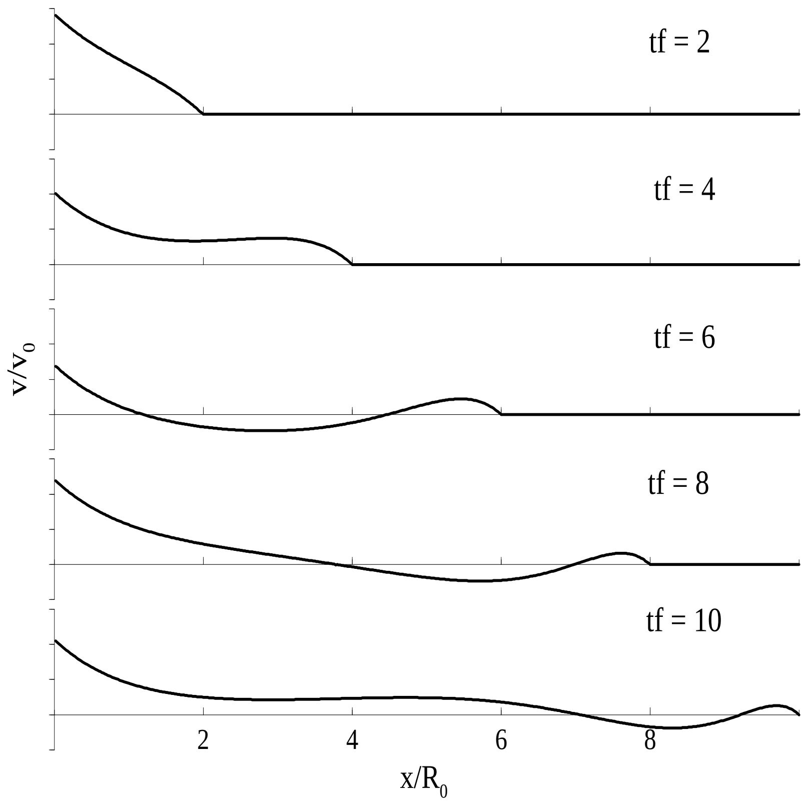


Figure 5.5. v-componente in response to an initial step-like elevation at $x = 0$

ξ and v can be calculated,

$$
\begin{aligned}
v(xt) &= c\frac{\xi_0}{H}\theta(ct-|x|)\int_{\frac{|x|}{c}}^{t} dt' f J_0\left(f\sqrt{t'^2-\frac{x^2}{c^2}}\right), && (5.76)\\
\xi(xt) &= \xi_0(x)\,(1-\theta(ct-|x|))\\
&\quad +\xi_0\theta(ct-|x|)\int_{\frac{|x|}{c}}^{t} dt' \frac{fx}{c\sqrt{t'^2-\frac{x^2}{c^2}}} J_1\left(f\sqrt{t'^2-\frac{x^2}{c^2}}\right). && (5.76)
\end{aligned}
$$

The remaining time integral must be calculated numerically in the general case.

We compare our solutions with those of the non-rotating case. In the limit $f \approx 0$ we gain the result for the non-rotating case from

$$
J_0(0) = 1, \quad J_1(0) = 0. \tag{5.77}
$$

i.e.,

$$\begin{aligned}
\xi(xt) &= \xi_0(x)\,(1-\theta(ct-|x|),) \\
u(xt) &= -c\frac{\xi_0}{H}\theta(ct-|x|), \\
v(x,t) &= 0.
\end{aligned} \tag{5.76}$$

There are wave fronts spreading to both direction with the speed c. Hence, c is the maximum wave speed for small wave number or large wave length. In the non-rotating case the waves are non-dispersive. All wave fronts arrive at the same time and the flow is adjusted and steady after the wave frontw has passed through. In the rotating case the waves are dispersive. After the very long waves have passed through, the sea level becomes suddenly zero, u is rapidly accelerated. Now step by step the shorter waves arrive and modify the flow and the sea level. For large times, $tf \gg \frac{|x|}{R_0}$, the asymptotic representation of the Bessel function,

$$J_0(z) \approx \sqrt{\frac{2}{\pi z}}\cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right), \tag{5.77}$$

can be used. For u we get the asymptotic result

$$u(xt) \approx -c\frac{\xi_0}{H}\sqrt{\frac{2}{\pi ft}}\cos\left(ft-\frac{\pi}{4}\right), \tag{5.78}$$

i.e., it changes its sign periodically, but with an algebraically decreasing amplitude. Similar terms but with different phase are found for v and ξ. Asymptotically for $t\to\infty$ the integrals can be carried out,

$$v \to c\frac{\xi_0}{H}\mathrm{e}^{-\frac{|x|}{R_0}}, \tag{5.79}$$

$$\xi \to \xi_0\mathrm{sgn}(x)\left(1-\mathrm{e}^{-\frac{|x|}{R_0}}\right). \tag{5.80}$$

This is the steady state result gained previously for the conservation of potential vorticity.

Here we see again a basic principle of geostrophic adjustment. Waves radiating away from an perturbation leaving behind a geostrophically balanced flow and sea level pattern. The dynamic of the waves is governed by the conservation of potential vorticity. The response at a fixed position depends on the waves dispersion relation, hence of the phase speed in dependence on the wave number.

Finally an inconsistency within the theory needs to be mentioned. Usually, the ocean depth exceeds 1 km and the Rossby radius amounts about 1000 km. Hence, the f-plane approximation is not valid in detail. Nevertheless, the examples show a basic principle which keeps its physical meaning, when a zonal variability of the Coriolis force makes the response pattern more complex.

5.2. Waves in a channel

5.2.1. A meridionally oriented channel, fplane approximation

We investigate, which kind of waves may spread through a channel. Our channel is placed on a rotating planet, for a moment we consider the f-plane case, i.e., the Coriolis parameter is assumed to be a constant in the area under consideration. To become familiar with the wave types, we start with the shallow water case. Considering free waves, i.e., the equations to start with are

$$U_t - fV + gH\xi_x = 0, \tag{5.81}$$
$$V_t + fU + gH\xi_y = 0, \tag{5.82}$$
$$\xi_t + U_x + V_y = 0. \tag{5.83}$$

The co-ordinate system may be placed in the center of the channel. The channel is elongated and has steep boundaries at $x = -b$ and $x = b$. Here the cross shore velocity vanishes,

$$u = 0 \quad \text{for} \quad x = \pm b. \tag{5.84}$$

The boundary condition implies a special balance between the long-shore current V and the cross-shore pressure gradient,

$$-fV + gH\xi_x = 0 \quad \text{for} \quad x = \pm b. \tag{5.85}$$

At the coasts, the long-shore acceleration is only due to the long-shore pressure gradient, since the Coriolis acceleration due to the cross-shore flow vanishes,

$$V_t + gH\xi_y = 0 \quad \text{for} \quad x = \pm b, \tag{5.86}$$

For the sea level elevation alone we get

$$f\xi_y + \xi_{xt} = 0 \quad \text{for} \quad x = \pm b. \tag{5.87}$$

As the next step, we derive an equation for either the sea level or the cross-channel velocity alone. To eliminate the sea level and the long-channel velocity we differentiate the first equation with respect to time and substituting V_t we get

$$U_{ttt} + f^2 U_t + fgH\xi_{yt} + gH\xi_{xtt} = 0. \tag{5.88}$$

With the sea level equation we can substitute ξ,

$$U_{ttt} + f^2 U_t - fgH(U_{xy} + Vyy) - gH(U_{xxt} + Vyxt) = 0. \tag{5.89}$$

From the V-equation and the U-equation respectively we find

$$V_{yxt} = -fU_{xy} - gH\xi_{yyx}, \tag{5.90}$$
$$-fVyy = U_{yyt} - gH\xi_{xyy}. \tag{5.91}$$

Hence, the terms with V cancel and we get a single equation for U,

$$\frac{\partial}{\partial t}\left(U_{tt} + f^2 U - gH\Delta U\right) = 0. \quad (5.92)$$

A similar equation can be derived for V. Alternatively, we may eliminate the velocities, and find

$$\frac{\partial}{\partial t}\left(\xi_{tt} + f^2 \xi - gH\Delta \xi\right) = 0. \quad (5.93)$$

We recognize the phase speed of shallow water waves in the non-rotating case, $c^2 = gH$, the equations permit a wave-like solution. Nevertheless, there is also a steady solution of a non-divergent flow in geostrophic balance with the sea level. This leads us to the incomplete system of geostrophic equations,

$$-fV + gH\xi_x = 0, \quad (5.94)$$
$$+fU + gH\xi_y = 0, \quad (5.95)$$
$$U_x + V_y = 0. \quad (5.96)$$

A simple example consistent with the boundary conditions for U is

$$U = 0, \quad (5.97)$$
$$V = V(x), \quad (5.98)$$
$$\xi(x) - \xi(-b) = \frac{f}{c^2}\int_{-b}^{x} dx V(x). \quad (5.99)$$

This geostrophic relation is interesting in so far, that the sea level gradient across the channel is related to the total volume transport. This would allow to diagnose the total volume flux by measuring sea level at the coasts. This is really done in many cases, but it does not give deeper insight in the nature of the channel flow here.

The spectrum

To see the properties of the time dependent solution we make a Fourier ansatz

$$\gamma(xyt) = e^{-i\omega t + ily}\gamma(xl\omega), \quad (5.100)$$

where $\gamma = (\xi, U, V)$. We find an ordinary wave equation

$$i\omega\left(\alpha^2\gamma - \gamma_{xx}\right) = 0, \quad (5.101)$$
$$\alpha^2 = c^{-2}\left(\omega^2 - f^2\right) - l^2. \quad (5.102)$$

The general solution of Equation (5.102) for $\gamma = U$ is

$$U = A\sin(\alpha x) + B\cos(\alpha x), \quad (5.103)$$

from the boundary condition for U it follows

$$A\sin(\alpha b) + B\cos(\alpha b) = 0, \tag{5.104}$$
$$-A\sin(\alpha b) + B\cos(\alpha b) = 0. \tag{5.105}$$

A solution exists only if the coefficient determinate vanishes:

$$2AB = 0. \tag{5.106}$$

Hence, either A or B or both quantities must be zero.

•$A = 0$ The boundary condition limits α to the discrete values

$$\alpha_n b = \frac{(2n+1)\pi}{2}. \tag{5.107}$$

This limits the spectrum of ω and leads us to the dispersion relation

$$\alpha^2 = \alpha_n^2 \tag{5.108}$$

or equivalently

$$\omega^2 = f^2 + c^2\left(l^2 + \alpha_n^2\right). \tag{5.109}$$

The resulting solution is symmetric

$$U(xl\omega) = B\cos(\alpha_n x). \tag{5.110}$$

•$B = 0$ The boundary condition limits α to the discrete values

$$\alpha_n b = n\pi = \frac{2n\pi}{2}. \tag{5.111}$$

The dispersion relation is the same like in the previous case,

$$\omega^2 = f^2 + c^2\left(l^2 + \alpha_n^2\right), \tag{5.112}$$

but the resulting solution is anti-symmetric

$$U(xl\omega) = A\sin(\alpha_n x). \tag{5.113}$$

•$U = 0$ In the special case that both the coefficients vanish, i.e., $U = 0$, we get equations for V and ξ alone,

$$\begin{aligned} -fV + c^2\xi_x &= 0, && (5.114)\\ -i\omega V + ilc^2\xi &= 0, && (5.115)\\ -i\omega\xi + ilV &= 0. && (5.116)\end{aligned}$$

The last two equations are linearly dependent. Hence, a solution exists only for a vanishing coefficient determinant. This provides us with the dispersion relation

$$\omega^2 = c^2 l^2. \tag{5.117}$$

Hence, both the phase and the group velocity is independent of the wave number and amounts to

$$c_p = c_g = \frac{\omega}{l} = \pm c. \tag{5.118}$$

The positive sign corresponds to a wave following the direction of a coordinate, hence, spreading east- or north-ward. The negative sign appears for waves spreading south- or west-ward.

Poincaré waves

The symmetric and the anti-symmetric solution for $A = 0$ or $B = 0$ can be written together,

$$\omega^2 = f^2 + c^2\left(l^2 + \left(\frac{n\pi}{2b}\right)^2\right), \tag{5.119}$$

$$\begin{aligned} U &= A\sin\left(\frac{n\pi}{2b}x\right) \qquad n \quad \text{even}, && (5.120)\\ U &= A\cos\left(\frac{n\pi}{2b}x\right) \qquad n \quad \text{uneven}. && (5.121)\end{aligned}$$

Both, phase speed and group speed of these waves, depend on the wave number. The waves are dispersive. With

$$\begin{aligned} \omega &= \pm c\sqrt{k^2 + \frac{f^2}{c^2}},\\ k^2 &= l^2 + \left(\frac{n\pi}{2b}\right)^2. && (5.121)\end{aligned}$$

If the wave number is large we get the dispersion relation for the non-ratating shallow ater waves. For long waves, i.e. small wave numbers, rotation of the earth matters and frequency and phase speed is enhanced compared with the non-rotating case. The inertial frequency is the minimum frequency for Poincaré waves. The dispersion relation may be rewritten like

$$\omega = \pm c\sqrt{k^2 + \frac{1}{R^2}}. \tag{5.122}$$

The *barotropic Rossby radius*

$$R = \frac{c}{|f|}, \tag{5.123}$$

is a typical minimum length scale for Poincaré waves. If the wave length exceeds the Rossby radius the dispersion relation is strongly influenced by rotation, otherwise it is not.

Kelvin waves

For the case $U = 0$ the remnant of the momentum equations can be used to determine ξ. We eliminate the velocities from the momentum equations and the continuity equation and get

$$\xi_{tt} - c^2\xi_{yy} = 0, \tag{5.124}$$

$$f\xi_y + \xi_{xt} = 0. \tag{5.125}$$

The first equation is a wave equation. We do not make a Fourier ansatz, the characteristic solution of the wave equation reads simply

$$\xi(x, y, t) = A(x)\bar{\xi}_0(y - ct) + B(x)\bar{\xi}_0(y + ct). \tag{5.126}$$

Inserting into the second equation gives a condition for A and B,

$$\left(fA - cA'\right)\bar{\xi}_0'(y - ct) + \left(fB + cB'\right)\bar{\xi}_0'(y + ct) = 0. \tag{5.127}$$

This equation must be valid for all arguments of $\bar{\xi}_0$ which implies that the coefficients must vanish separately, i.e.,

$$fA - cA' = 0., \tag{5.128}$$

$$fB + cB' = 0. \tag{5.129}$$

The solution is a simple exponential function,

$$A = A_0 e^{x/R\,\mathrm{sgn}(f)}, \tag{5.130}$$

$$B = B_0 e^{-x/R\,\mathrm{sgn}(f)}, \tag{5.131}$$

(5.132)

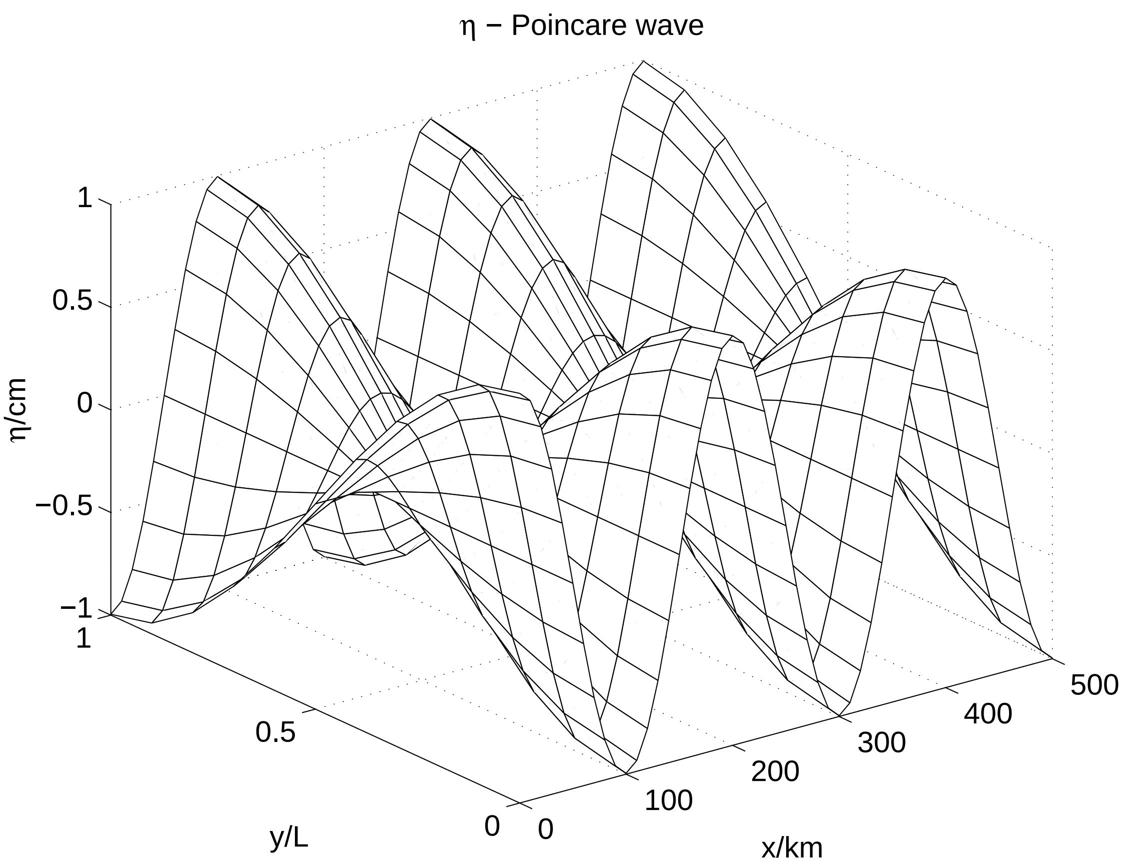


Figure 5.6. The undulating sea level for Poincaré waves. Note, the x- and y- coordinates are exchanged compared with the text. (Figure by W. Fennel)

The *barotropic Rossby radius*

$$R = \frac{c}{|f|}, \tag{5.133}$$

is here roughly the distance a wave spreads within one inertial period.

Now, we have to distinguish several cases, i.e., the northern and the southern hemisphere and varying wave direction represented by the sign of the phase speed. To see the basic principle of the wave dynamics we assume that the channel width exceeds the Rossby radius.

- At the northern hemisphere, i.e., for $\text{sgn}(f) = 1$, a wave spreading north-ward has positive phase speed, the sign of the exponential is "+", the wave is localized at the eastern coast of the channel. The amplitude declines exponentially to the west.

Introducing a constant factor makes this more transparent,

$$\xi \approx A_0\bar{\xi}_0(y-ct)e^{(x-b)/R}. \qquad (5.134)$$

If the dimension of the channel exceeds the Rossby radius, i.e., $b \gg R$, the wave amplitude is negligible at the western boundary.

- For a wave spreading south-ward the amplitude has its maximum at the western coast and declines rapidly eastward. Hence, this wave is localized at the western boundary of the channel,

$$\xi \approx B_0\bar{\xi}_0(y+ct)e^{-(x+b)/R}. \qquad (5.135)$$

- At the southern hemisphere sgn$(f) = -1$. Again, a wave spreading north-ward has positive phase speed. Now it is localized at the western boundary. Making a constant exponential factor explicit, we find

$$\xi \approx A_0\bar{\xi}_0(y-ct)e^{-(x+b)/R}. \qquad (5.136)$$

- A south-ward spreading wave is related to a negative phase speed. At the southern hemisphere the solution reads

$$\xi \approx B_0\bar{\xi}_0(y+ct)e^{(x-b)/R}. \qquad (5.137)$$

Those coastally trapped non-dispersive waves are called Kelvin waves. Figure 5.2.1 gives an overview. It shows some elements of the coastal wave guides of the ocean. Kelvin waves are spreading pole-ward at eastern coastlines and are spreading equator-ward at western coasts.

The meridional velocity is defined by the geostrophic balance with the sea level elevation,

$$\begin{aligned} v(xyt) &= f^{-1}c^2\xi_x \\ &= cA(x)\bar{\xi}_0(y-ct) - cB(x)\bar{\xi}_0(y+ct).. \end{aligned} \qquad (5.137)$$

In summary, considering an initial sea level elevation $\xi_0(x, y, t)$ part of it generates a wave moving with the coast to the right at the northern hemisphere and the coast to the left at the southern hemisphere. The propagating wave keeps the original along-channel shape of the sea level elevation. This is possible since the waves are non-dispersive, i.e., waves with all wave numbers move with the same phase speed and group speed as well. The wave amplitude is decreasing exponentially off-shore. Since it may be either positive or negative if may support pole-ward and equator-ward flow as well.

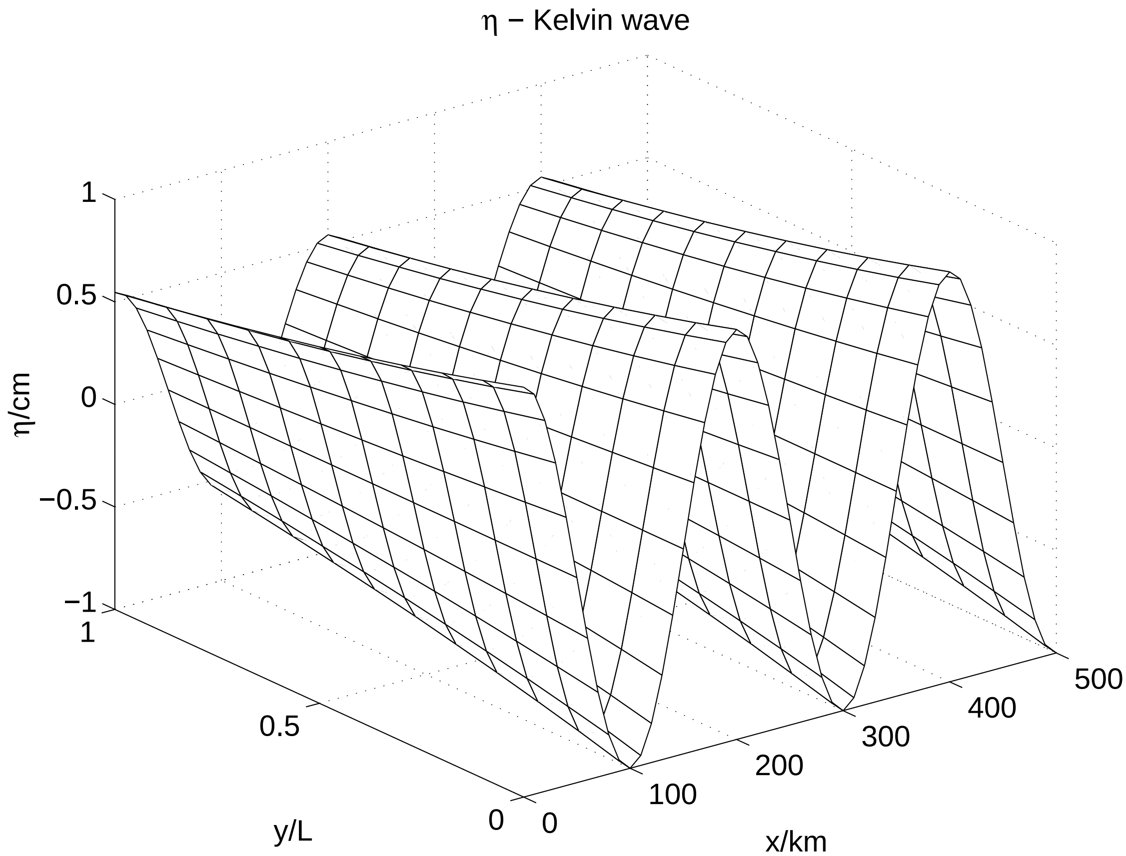


Figure 5.7. The undulating sea level for Kelvin waves. Note, the x- and y- coordinates are exchanged compared with the text. (Figure by W. Fennel)

It is possible to exchange the zonal velocity u and v, as long as the mathematically positive sense of the coordinate system is not violated. Hence, in the northern hemisphere Kelvin waves are spreading with the coast to the right and at the southern hemisphere with the coast to the left. This dependency on the Coriolis forces breaks symmetry of the oceanic wave system.

5.2.2. Cross channel sloping bottom

If the bottom of the channel is not flat, $c^2 = gH$ depends on the coordinates. Let us consider a cross-channel slope of the bottom,

$$H(x) = H_0 \left(1 - s\frac{x}{b}\right). \qquad (5.138)$$

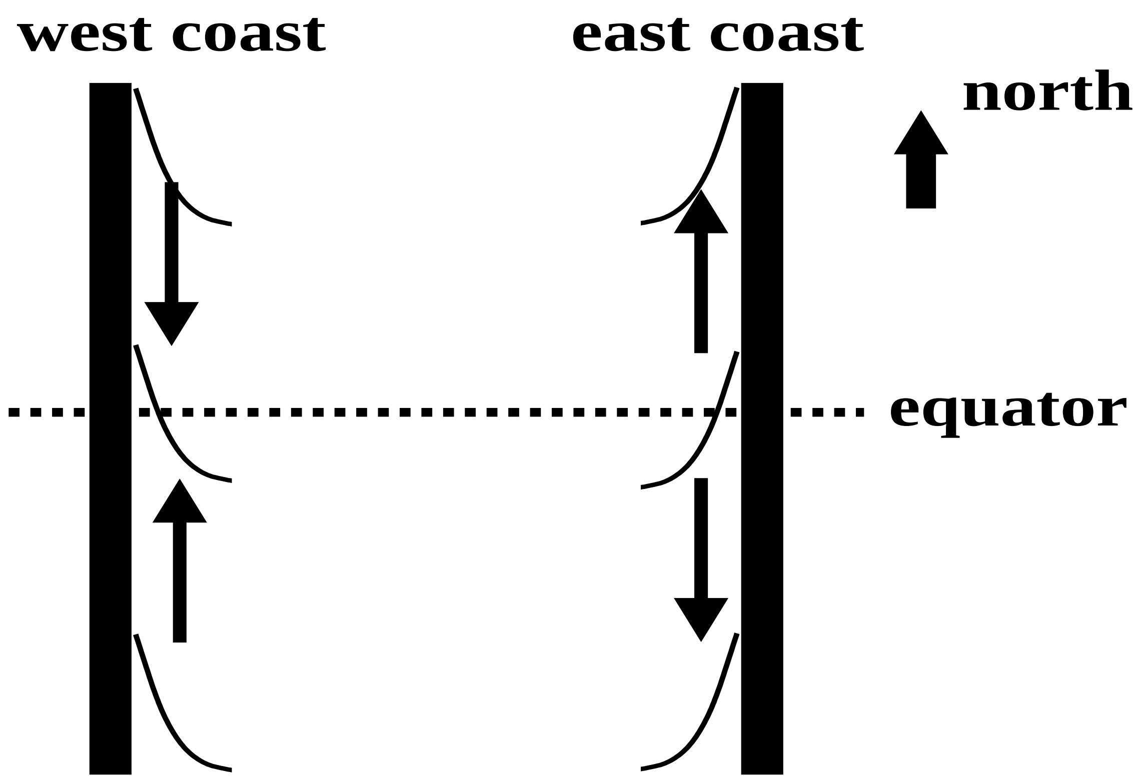


Figure 5.8. Coastlines as wave guides for Kelvin waves

We return to the more general sea level equation 5.48,

$$\frac{\partial}{\partial t}\left(\xi_{tt} + f^2\xi - g\boldsymbol{\nabla}_h \cdot (D\nabla_h\xi)\right) \;=\; gfJ(D,\xi). \tag{5.139}$$

We assume again $H \gg \xi$. For our special choice the Jacobi determinant reads

$$J(D,\xi) \;\approx\; J(H,\xi) = H_x\xi_y - H_y\xi_x = -s\frac{H_0}{b}\xi_y. \tag{5.140}$$

The term $g\boldsymbol{\nabla}_h \cdot (D\boldsymbol{\nabla}_h\xi)$ becomes

$$g\boldsymbol{\nabla}_h \cdot (D\boldsymbol{\nabla}_h\xi) \;\approx\; gH\Delta\xi - s\frac{gH_0}{b}\xi_x \tag{5.141}$$

Looking for free waves, the periodic ansatz may be useful,

$$\xi = \xi_0 e^{-i(\omega t - ky)}\bar{\xi}(x). \tag{5.142}$$

The resulting equation for ξ reads now,

$$-i\omega\left(\left(-\omega^2 + f^2 + gHl^2\right)\bar{\xi} - gH\bar{\xi}_{xx} + s\frac{gH_0}{b}\bar{\xi}_x\right) + il\,fgH_0\frac{s}{b}\bar{\xi} \;=\; 0. \tag{5.143}$$

Dividing by $-i\omega$ and $-gH_0$ we find

$$\left(1 - s\frac{x}{b}\right)\bar{\xi}_{xx} - \frac{s}{b}\bar{\xi}_x + \left(\frac{\omega^2 - f^2}{gH_0} - \left(1 - s\frac{x}{b}\right)l^2 + \frac{l}{\omega}\frac{sf}{b}\right)\bar{\xi} = 0. \tag{5.144}$$

As a simplification we assume that the elevation of the bottom is small compared to the total depth. Hence, we can restrict the theory to terms of first order in the bottom elevation,

$$\bar{\xi}_{xx} - \frac{s}{b}\bar{\xi}_x + \left(\frac{\omega^2 - f^2}{gH_0} - l^2 + \frac{l}{\omega}\frac{sf}{b}\right)\bar{\xi} = 0. \tag{5.145}$$

For short notation we define

$$\bar{\xi}_{xx} - p\bar{\xi}_x + q^2\bar{\xi} = , \tag{5.146}$$

$$p = \frac{s}{b}, \tag{5.147}$$

$$q^2 = \left(\frac{\omega^2 - f^2}{gH_0} - l^2 + \frac{l}{\omega}\frac{sf}{b}\right). \tag{5.148}$$

For $s = 0$ we get the well known wave equation for channel waves, but now in terms of the sea level. The boundary condition, $U = 0$ for $x = \pm b$ reads

$$-fV + gH\xi_x = 0 \quad \text{for} \quad x = \pm b, \tag{5.149}$$

$$V_t + gH\xi_y = 0 \quad \text{for} \quad x = \pm b, \tag{5.150}$$

or equivalently in the Fourier space

$$lf\bar{\xi} - \omega\bar{\xi}_x = 0 \quad \text{for} \quad x = \pm b. \tag{5.151}$$

The general solution for $\bar{\xi}$ reads

$$\bar{\xi} \sim e^{i\Lambda x} \tag{5.152}$$

and the resulting characteristic equation is

$$-\Lambda^2 - ip\Lambda + q^2 = 0, \tag{5.153}$$

$$\Lambda_{12} = -i\frac{p}{2} \pm \sqrt{-\frac{p^2}{4} + q^2}. \tag{5.154}$$

Hence the principle solution must have the general form

$$\bar{\xi} \sim e^{\frac{sx}{2b}} e^{\pm i\alpha x}, \tag{5.155}$$

$$\alpha^2 = q^2 - \frac{p^2}{4}, \tag{5.156}$$

$$= \left(\frac{\omega^2 - f^2}{gH_0} - l^2 + \frac{l}{\omega}\frac{sf}{b}\right) - \frac{s^2}{4b^2} \tag{5.157}$$

As the next step we must build a special solution that fulfills the boundary condition at the coast,

$$\bar{\xi} = e^{\frac{sx}{2b}}\left(Ae^{i\alpha x} + Be^{-i\alpha x}\right). \tag{5.158}$$

and consequently,

$$A\left(\omega\left(\frac{s}{2b} + i\alpha\right) - lf\right)e^{\frac{s}{2}}e^{i\alpha b} + B\left(\omega\left(\frac{s}{2b} - i\alpha\right) - lf\right)e^{\frac{s}{2}}e^{-i\alpha b} = 0, \tag{5.159}$$

$$A\left(\omega\left(\frac{s}{2b} + i\alpha\right) - lf\right)e^{-\frac{s}{2}}e^{-i\alpha b} + B\left(\omega\left(\frac{s}{2b} - i\alpha\right) - lf\right)e^{-\frac{s}{2}}e^{i\alpha b} = 0, \tag{5.160}$$

(5.161)

The coefficient determinant is

$$\left(\left(\frac{s}{2b} - \frac{fl}{\omega}\right)^2 + \alpha^2\right)\sin 2\alpha b = 0. \tag{5.162}$$

In the bracket all terms with s cancel out exactly,

$$\left(\left(\frac{s}{2b} - \frac{fl}{\omega}\right)^2 + \alpha^2\right) = \left(\frac{fl}{\omega}\right)^2 + \frac{\omega^2 - f^2}{gH_0} - l^2, \tag{5.163}$$

$$= \frac{\omega^2 - f^2}{gH_0\omega^2}\left(\omega^2 - gH_0 l^2\right). \tag{5.164}$$

Hence, to the first order the Kelvin wave frequency is unchanged, only the value of α for the Kelvin wave frequency is modified from the bottom slope,

$$\alpha^2(\omega = \pm l\sqrt{gH_0}) = \left(-\frac{f^2}{gH_0} \pm \frac{1}{\sqrt{gH_0}}\frac{sf}{b}\right) - \frac{s^2}{4b^2}, \tag{5.165}$$

$$= -\left(\frac{1}{R_0} \mp \frac{s}{2b}\right)^2. \tag{5.166}$$

Without a proof we give the result for ξ at the northern hemisphere for an eastern boundary,

$$\xi(x, y, t) = Ae^{(x-b)/R}\bar{\xi}_0(y - ct). \tag{5.167}$$

Within the accuracy of s/b a coastal Kelvin wave is not influence by a sloping sea floor.

Now we consider the roots related to

$$\sin 2\alpha b = 0. \tag{5.168}$$

For a flat bottom those led us to the Poincaré waves. Again, we get

$$\alpha^2 = \alpha_n^2 = \frac{n^2\pi^2}{4b^2}. \tag{5.169}$$

but from the definition of α^2 we find

$$\omega^2 - f^2 - c^2\left(l^2 + \alpha_n^2 + \frac{s^2}{4b^2}\right) + \frac{lsfc^2}{\omega b} \quad = \quad 0. \tag{5.170}$$

For a zero slope of the bottom, $s = 0$, we get again the dispersion relation of Poincaré waves.

Now we distinguish two limiting cases,

- $\omega^2 \gg f^2$ which is the case for large wave numbers. In this case the last term should be small, and we find a slightly modified dispersion relation for the Poincaré waves.
- $\omega^2 \ll f^2$ which becomes possible for the dispersion relation with sloping bottom. Considering only the leading terms we find

$$\omega \quad = \quad \frac{\frac{sf}{b}l}{l^2 + \alpha_n^2 + R^{-2}}. \tag{5.171}$$

Waves with this type of dispersion relation are called Rossby waves. The mechanism behind these waves is the conservation of potential vorticity. Moving a fluid parcel to an area with another depth results in enhanced or reduced absolut vorticity since potential vorticity stays constant. Our analysis has shown that this response is wave like. These vorticity waves may exist only on a rotating planet, they disappear for $f = 0$.

At the northern hemisphere the phase velocity is positive for a positive slope parameter. In our example the phase speed is directed north-ward. It is decreasing monotonically with increasing wave number l. The group velocity is more complex

$$\frac{\partial \omega}{\partial l} \quad = \quad \frac{sf}{b}\frac{1}{l^2 + \alpha_n^2 + R^{-2}}\left(1 - \frac{2l^2}{l^2 + \alpha_n^2 + R^{-2}}\right). \tag{5.172}$$

This means, that there is a maximum frequency corresponding to zero group velocity at,

$$\frac{2l^2}{l^2 + \alpha_n^2 + R^{-2}} \quad = \quad 1, \tag{5.173}$$

or

$$l_{max}^2 \quad = \quad \alpha_n^2 + R^{-2}, \omega_{max} = \frac{sf}{2b} l_{max}^{-1}. \tag{5.174}$$

This leads us to final relations for phase- and group speed

$$c_p \quad = \quad \frac{sf}{b}\frac{1}{l^2 + l_{max}^2}, \tag{5.175}$$

$$c_g \quad = \quad \frac{sf}{b}\frac{l_{max}^2 - l^2}{\left(l^2 + l_{max}^2\right)^2} \tag{5.176}$$

For vorticity waves with large wave numbers the spreading of signals and energy may happen in opposite direction.

Without a proof we give the the final result for sea level elevation and velocities. The zero-order approximation in s reads

$$\xi = A\sin\alpha_n x\cos(ly-\omega t)+\mathcal{O}(s), \tag{5.177}$$

$$U = \frac{c^2 l}{f}\sin\alpha_n x\sin(ly-\omega t)+\mathcal{O}(s), \tag{5.178}$$

$$V = \frac{c^2\alpha_n}{f}\cos\alpha_n x\cos(ly-\omega t)+\mathcal{O}(s). \tag{5.179}$$

Hence, on a rotating planet vorticity waves can be formed. They are approximately geostrophically balanced sea level elevations changing with very low frequency and spreading with low wave speed. Phases are following lines of constant depth leaving the shallower area to the right.

5.3. Waves on the β-plane

Now we consider another example. The Coriolis parameter f varies meridionally, i.e., at a latitude φ_0 like

$$f(\varphi) = 2\Omega\sin\varphi \approx f(\varphi_0)+\beta y = f_0+\beta y. \tag{5.180}$$

The parameter β describes the variability of the Coriolis parameter in linear approximation,

$$\beta = \frac{2\Omega\cos\varphi_0}{R} \tag{5.181}$$

has its maximum near the equator and vanishs at the poles. R is the radius of the earth.

$$U_t - fV + gH\xi_x = 0, \tag{5.182}$$

$$V_t + fU + gH\xi_y = 0, \tag{5.183}$$

$$\xi_t + U_x + V_y = 0. \tag{5.184}$$

These equations can be combined to an equation in V alone,

$$\frac{\partial}{\partial t}\left(V_{tt}+f^2V-c^2\Delta V\right)-c^2\beta V_x = 0. \tag{5.185}$$

Exercise 4.20 Derive this equation!

From the variability of the Coriolis parameter there arises a new contribution in the wave equations. It breaks the symmetry between eastern and northern direction, a channel that orients north-ward differs from an east-ward directed channel. For a zonally oriented channel we get the boundary condition

$$V = 0 \qquad \text{for} \quad y = \pm a, \tag{5.186}$$

for a meridional channel we find from $U = 0$,

$$-fV_y - \beta V - V_{tx} = \qquad \text{for} \quad x = \pm b. \tag{5.187}$$

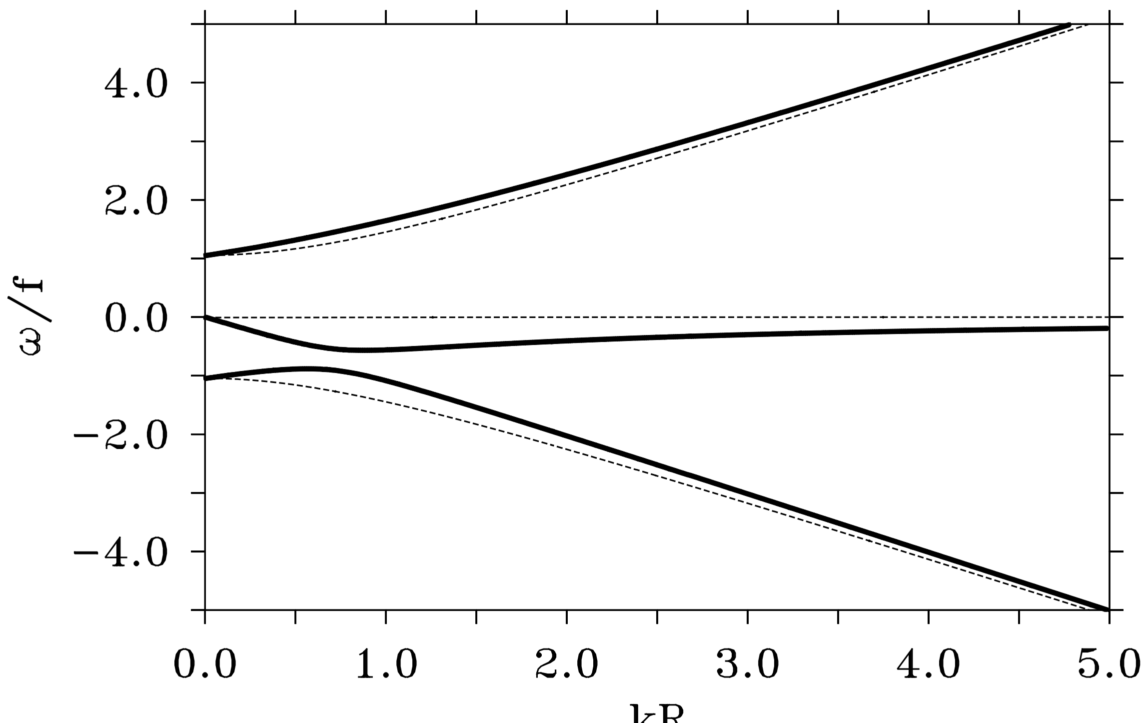


Figure 5.9. The dispersion relation of free waves in the open ocean. The nearly parabolic branches correspond to Poincaré waves, the other branch to Rossby waves. The dashed lines depict the f-plane approximation.

To study the properties of the wave-like solution a periodic ansatz is not appropriate. The resulting dispersion relation would contain expresions with f, but f depends on y. Hence, the frequency ω would depend on y in contradiction to the initial assumptions. Instead we transform time and zonal variable only and get an eigenvalue problem of the form

$$-i\omega\left(-\omega^2 + f^2 + c^2k^2\right)V(\omega,k,y) - ic^2k\beta V(\omega,k,y) - c^2V_{yy}(\omega,k,y) = 0 \tag{5.188}$$

Eventually, for a meridional channel the spectrum of k is limited to discrete values. We may rewrite this equation like

$$V_{yy} + g(y)V + l^2 V = 0. \tag{5.189}$$

This is a Sturm-Liouville eigenvalue problem that is known to have a countable infinite number of eigenvalues, l^2.

To find an approximate solution we linearize the terms with f,

$$f \approx f_0. \tag{5.190}$$

This leads us to the dispersion relation

$$-i\omega\left(-\omega^2 + f_0^2 + c^2(k^2 + l^2)\right) - ic^2 k\beta = 0. \tag{5.191}$$

For $\beta = 0$ we find the f-plane dispersion relation for Poincaré waves. Figure 5.9 shows the three brances of the numerical solution.

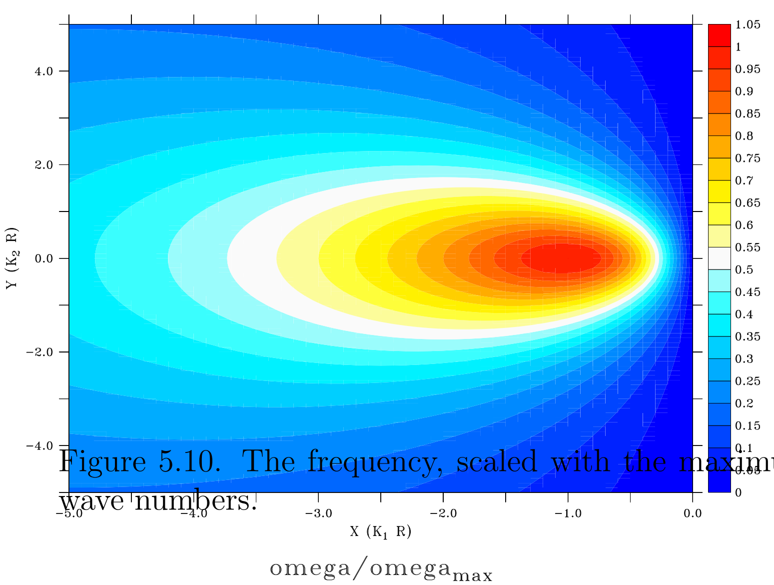


Figure 5.10. The frequency, scaled with the maximum frequency in dependency on the wave numbers.

The Rossby wave dispersion relation

Rossby waves have a small frequency. This allows to derive an analytical expression for the Rossby wave branch alone. For $|\omega| \ll |f|$ one finds

$$\omega = -\frac{c^2 k\beta}{f_0^2 + c^2(k^2+l^2)}, \tag{5.192}$$

$$= -\frac{kR\,\beta R}{1 + R^2(k^2+l^2)} \tag{5.193}$$

$$. \tag{5.194}$$

The phase speed of Rossby waves is negative, hence, these waves are propagating west-ward and south-ward. To investigate the analytical properties of the group velocity we consider a fixed l. There maximum of ω is found from

$$\frac{\partial\omega}{\partial k} = -\frac{\beta}{1+(k^2+l^2)R^2} + \frac{2k^2\beta}{(1+(k^2+l^2)R^2)^2} = 0. \quad k = -\sqrt{1+l^2R^2}. \tag{5.195}$$

The maximum frequency is

$$\omega_m = +\frac{\beta R^2}{2\sqrt{1+l^2R^2}}. \tag{5.196}$$

The global maximum is found for $l = 0$,

$$\omega_{max} = +\frac{\beta R}{2}. \tag{5.197}$$

If there is a maximum frequency in dependency on k, the group velocity changes change its sign here. The group velocity vector is,

$$c_g = \begin{pmatrix} c_g^x \\ c_g^y \end{pmatrix} = \begin{pmatrix} \frac{\beta\left((k^2-l^2)R^2-1\right)}{((k^2+l^2)R^2+1)^2} \\ \frac{2\beta k l R^2}{((k^2+l^2)R^2+1)^2} \end{pmatrix} \tag{5.198}$$

Hence, the energy of small wave number Rossby waves (long waves) propagates west-ward. For large wave numbers (short waves) the energy propagates east-ward.

Potential vorticity

To understand the wave dynamics better, it helps to consider the conservation of potential vorticity. To this end we rewrite the dynamic equations in terms of relative vorticity $\chi = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u$ and sea level ξ. The rotation of the momentum equations is

$$\frac{\partial}{\partial t}(V_x - U_y) + f(U_x + V_y) + \beta V = 0, \tag{5.199}$$

or

$$\frac{\partial}{\partial t}\left(\frac{H\chi}{f} - \xi\right) + \frac{\beta}{f}V = 0, \tag{5.200}$$

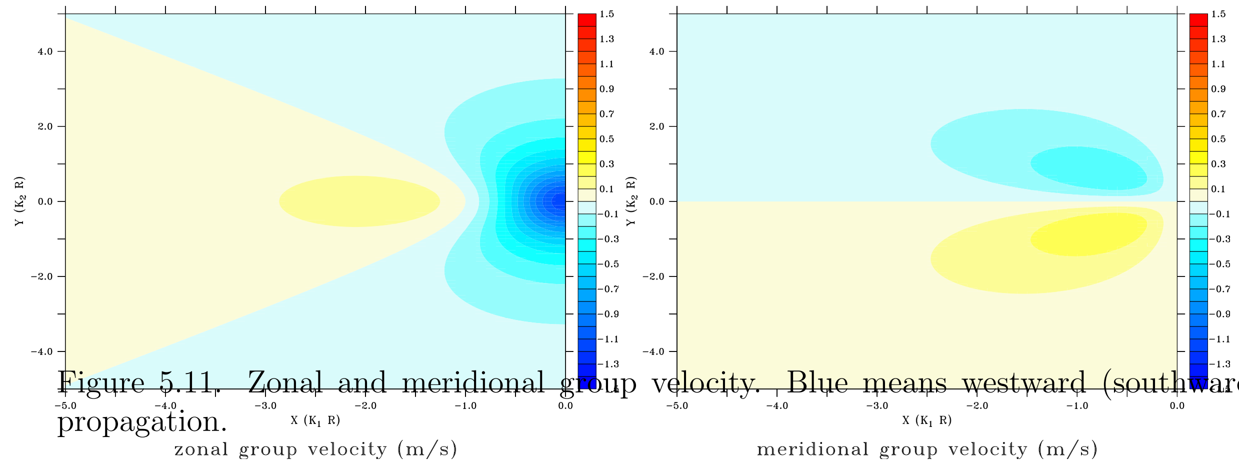


Figure 5.11. Zonal and meridional group velocity. Blue means westward (southward) propagation.

This is the linearized form of the potential vorticity conservation equation 5.39. Considering velocity scales of 10^{-2}ms^{-1}, mid-latitudes, horizontal scales of 1000 km and ocean depth of 5000 m, the sea level elevation is small compared with the vorticity term, i.e.,

$$\frac{\partial}{\partial t}\chi + \beta v \approx 0, \tag{5.201}$$

To understand the restoring force we consider an idealized case of the conservation of angular momentum on the rotating earth, no forcing, no zonal flow, no vertical advection, linearized equations (Olbers et al. (2012), p 227)

$$\frac{\partial}{\partial t}\chi + \beta v = 0, \tag{5.202}$$

$$\chi = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \tag{5.203}$$

Northward moving water means increasing planetary vorticity f. Conservation of vorticity implies decreasing relative vorticity χ. Hence the flow shear v_x must be decreasing. The faster the northward flow the stronger its reduction. We consider an initial flow field directed northward with

$$v_0(x) = V\sin(x/L), \qquad \chi_0(x) = \frac{V}{L}\cos(x/L). \tag{5.204}$$

Integrating over time gives

$$\chi = \chi_0(x) - t\beta v_0(x) = \frac{V}{L}\cos(x/L) - t\beta V\sin(x/L) \approx \frac{V}{L}\cos\left(\frac{x}{L} + \beta L t\right) \tag{5.205}$$

Integrating over x gives v oscillating with a sin-function.

$$v \approx V \sin\left(\frac{x}{L} + \beta L t\right) \tag{5.206}$$

Remarkably the wave has the shape of the initial disturbance spreading westward, since $\beta L > 0$. It moves faster near the equator and slowly at high latitudes. Any initial rotational flow, for example a coastal jet or an undercurrent within the eastern boundary current system has the tendency to decay slowly by radiation of planetary waves. The phase speed depends on the horizontal scale L of the flow pattern, the planetary waves are dispersive.

Exercise 4.?? Discuss the similarity and differences of planetary and topographic Rossby waves!

Exercise 4.?? Discuss the time scales of planetary Rossby wave spreading!

5.4. Equatorial waves

Near the equator the Coriolis force becomes negligibly small. So one may expect that the cross wise coupling of meridional and zonal motion by the Coriolis acceleration does not play any role here. Indeed, one may assume a simple balance: In the surface layer zonal wind stress accelerates a zonal current. Since there are continents intersecting this current, pressure gradients build up. In the surface layer the wind stress becomes balanced by a gradient of the sea surface heigt η,

$$\rho_1 g \frac{\partial}{\partial x}\eta = \rho_s X, \tag{5.207}$$

an east wind (blowing westward, negative X) is balanced by a negative (high sea level in the west) sea surface elevation gradient. Below the wind driven surface layer the surface pressure gradient is compensated by an opposite gradient of the thermocline,

$$\rho_1 g \frac{\partial}{\partial x}\eta = (\rho_2 - \rho_1)\, g \frac{\partial}{\partial x} h. \tag{5.208}$$

As a result the acceleration becomes small throughout the water column. Indeed, measurements of temperature and salinity in the equatorial area show westward enhanced mixed layer depth and westward enhanced sea level elevation. However, this simplistic view does not show, how this balance is established. Morevoer, it does not reveal any information on the currents except that the acceleration is zero.

We have seen that the zonal variation of the Coriolis parameter permits wave like current patterns from the conservation of potential vorticity. This conservation is also valid at the equator with zero Coriolis force, any zonal movement corresponds to changing relative (local) vorticity. To understand these waves we start here a detailed analysis of the dispersion relation of equatorial waves and the corresponding wind driven flow pattern.

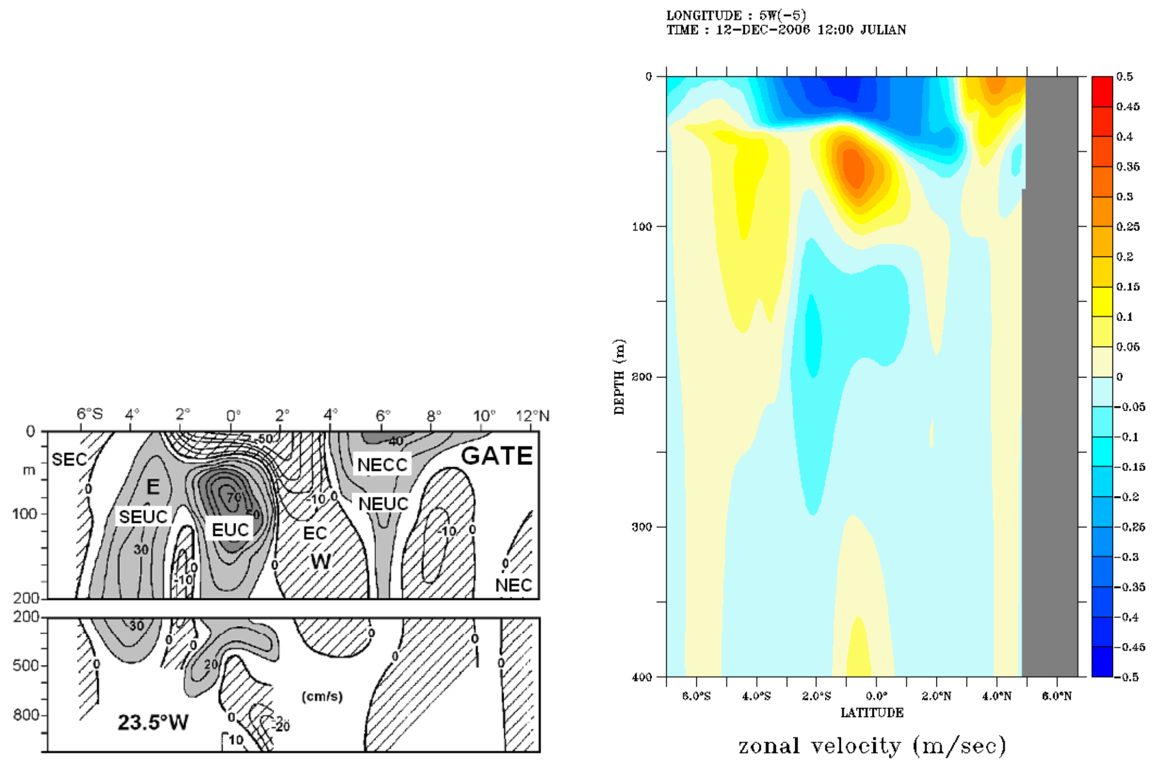


Figure 5.12. The equatorial currents found during the GATE experiment in the 60th (left) and seen in a numerical simulation (right).

$$\frac{\partial}{\partial t}\left(V_{tt} + (\beta y)^2 V - c^2 \Delta V\right) - c^2 \beta V_x = 0. \tag{5.209}$$

A periodic ansatz leads to

$$-i\omega\left(-\omega^2 + (\beta y)^2 + c^2k^2\right)V(\omega,k,y) - ic^2k\beta V(\omega,k,y) - c^2 V_{yy}(\omega,k,y) = 0 \tag{5.210}$$

We introduce dimensionless coordinates

$$s = \sqrt{\frac{\beta}{c}}y. \tag{5.211}$$

This leads us to the equation

$$v_{ss} + \left(2q + 1 - s^2\right)v = 0, \tag{5.212}$$

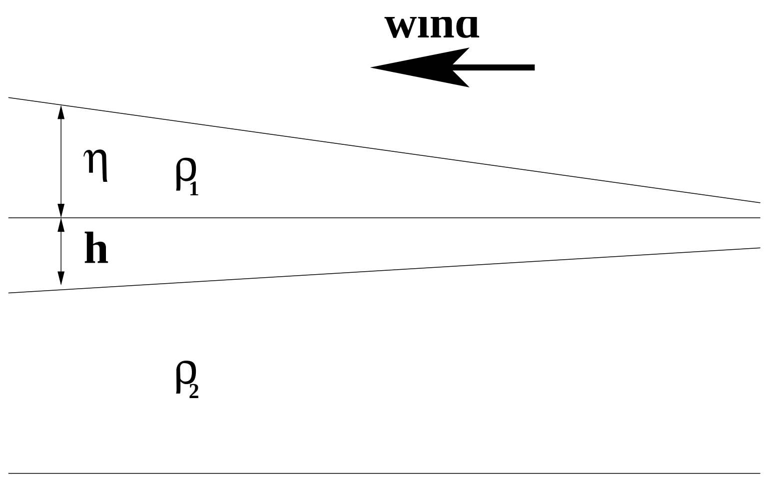


Figure 5.13. The wind induced surface elevation and the thermocline elevation.

with the abbreviation

$$2q + 1 = \frac{\omega^2 - c^2k^2}{\beta c} - \frac{ck}{\omega}. \quad (5.213)$$

The propertiea of this equations ar well known from the theory of the quantum mechanical harmonic oscillator. A solution finite a large distance from the origin exists only for integer values of q, i.e.,

$$q = m, \qquad m = 0, 1, 2, 3, ... \quad (5.214)$$

Remarkably, there is also a solution for $m = -1$ that however needs a separate consideration.

The solutions are Hermite functions $\psi_m(s)$,

$$\psi_m(s) = \frac{1}{\sqrt{2^m m! \sqrt{\pi}}} e^{-\frac{s^2}{2}} H_m(s), \quad (5.215)$$

where H_m are Hermite polynominals

$$H_0 = 1, H_1 = 2x, H_2 = 4x^2 - 2, H_3 = 8x^3 - 12x... \quad (5.216)$$

So we have found two important characteristics of the solution,

$$v = \sum_m a_m v_m, \quad (5.217)$$

$$v_m = e^{i(kx - \omega t)} \psi_m(s). \quad (5.218)$$

It is confined to the equator, a typical horizontal scale is the equatorial Rossby radius

$$R_0 = \sqrt{\frac{c}{\beta}}. \tag{5.219}$$

Hence, the equatorial ocean serves as a wave guide. The dispersion relation for the equatorial waves reads

$$2m+1 = \frac{\omega^2 - c^2k^2}{\beta c} - \frac{ck}{\omega}. \tag{5.220}$$

Two approximations gives a rough overview over the properties of the spectrum. For large values of m or k the frequency if high and we find

$$\omega^2 \approx (2m+1)c\beta + c^2k^2. \tag{5.221}$$

These waves have similar properties like Poincaré waves. For very small frequency we find the equatorial Rossby wave

$$\omega \approx -\frac{\beta ckc}{(2m+1)\beta c + c^2k^2}. \tag{5.222}$$

For a more detailed discussion of the spectrum we resolve the dispersion relation for the wave number

$$k_m^{1,2} = -\frac{\beta}{2\omega} \pm \frac{1}{c\omega}\sqrt{\frac{\beta^2c^2}{4} + \omega^4 - \beta c\omega^2(2m+1)}. \tag{5.223}$$

The analytical properties of the term under the square root are esential wether or not a real solution exists. First we ask for the zeros of the polynominal under the square root expression, i.e.,

$$\omega^4 + \frac{\beta^2c^2}{4} - \beta c\,\omega^2\,(2m+1) = 0. \tag{5.224}$$

The roots are

$$\begin{aligned}
\omega_{i,r}^2 &= \frac{\beta}{2}c\,(2m+1) \pm \sqrt{\frac{\beta^2}{4}c^2(2m+1)^2 - \frac{\beta^2c^2}{4}}, \\
&= \frac{\beta c}{2}\left(2m+1 \pm \sqrt{(2m+1)^2-1}\right), \\
&= \beta c\left(m \pm \sqrt{m(m+1) + \frac{1}{2}}\right).
\end{aligned} \tag{5.223}$$

This can be rewritten with

$$m \pm \sqrt{m(m+1)} + \frac{1}{2} = \left(\sqrt{\frac{m}{2}} \pm \sqrt{\frac{m+1}{2}}\right)^2 \tag{5.224}$$

as

$$\omega_{i,r} = \sqrt{\beta c}\left(\sqrt{\frac{m+1}{2}} \pm \sqrt{\frac{m}{2}}\right). \tag{5.225}$$

We assume that i corresponds to the $+$ and r corresponds to the $-$. With Vieta's theorem we can rewrite every polynominal in terms of it zeros and find

$$k_m^{1,2} = -\frac{\beta}{2\omega} \pm \frac{1}{c\omega}\sqrt{\left(\omega^2-\omega_i^2\right)\left(\omega^2-\omega_r^2\right)}. \tag{5.226}$$

Figure 5.14 show the dispersion relation. The frequency is scales with $\sqrt{\beta c}$, the wave number with $\sqrt{\beta/c}$.

For $m = 0$ both frequencies ω_i and ω_r coincide,

$$\omega_{i,r} = \sqrt{\beta c}\sqrt{\frac{1}{2}}, \tag{5.227}$$

and we find

$$k_0^{1,2} = -\frac{\beta}{2\omega} \pm \frac{1}{c\omega}\left(\omega^2 - \frac{\beta c}{2}\right). \tag{5.228}$$

The roots are

$$k_0^{(1)} = \frac{\omega}{c} - \frac{\beta}{\omega} \qquad \text{and} \qquad k_0^{(2)} = -\frac{\omega}{c}. < \tag{5.229}$$

The first root, $k_0^{(1)}$, corresponds to the so-called Yanai-wave. For low frequency it behves like a west-ward propagating Rossby wave, for high frequency it is approximately an inertial wave. The second root, $k_0^{(2)}$, corresponds to a spurious solution with zero amplitude for all variables.

For $m > 0$ both frequencies ω_i and ω_r are different. Hence, between ω_r and ω_i there is a gap in the frequency spectrum, where no real dispersion relation exists. We have shown above that for $\omega < \omega_r$ the waves behave like west-ward propagating Rossby waves. For $\omega > \omega_i$ the waves are similar to Poincaré waves.

Inserting the solution for V into the shallow water equations gives the zonal velocity U and the sea level ξ,

$$U = ic\sqrt{\frac{\beta}{c}}\frac{\omega s V - kcV_s}{\omega^2 - k^2c^2} \tag{5.230}$$

$$\xi = i\sqrt{\frac{\beta}{c}}\frac{kscV - \omega V_s}{\omega^2 - k^2c^2} \tag{5.231}$$

$$\tag{5.232}$$

Like V both quatities are non-zero only within a small band round near the equator defined by the equatorial Rossby radius.

It is obvious, that the case $\omega = kc$ requires special attention. To get finite values for U and ξ, V must be zero. This leads us to the system

$$U_t + gH\xi_x = 0, \tag{5.233}$$
$$+fU + gH\xi_y = 0, \tag{5.234}$$
$$\xi_t + U_x = 0. \tag{5.235}$$

This leads us to the wave equations

$$\begin{aligned} U_{tt} - c^2 U_{xx} &= 0, \\ \xi_{tt} - c^2 \xi_{xx} &= 0. \end{aligned} \tag{5.235}$$

The dispersion relation is obvious,

$$\omega^2 = c^2 k^2 \qquad \text{or} \qquad \omega = \pm ck. \tag{5.236}$$

From the second and third equation and the dispersion relation we find

$$sU = \mp U_s. \tag{5.237}$$

The upper sign corresponds to the upper sign in the dispersion relation. Hence, U must be of the form

$$U \sim e^{\mp \frac{s^2}{2}} \Pi(x \mp tc). \tag{5.238}$$

Clearly, the solution with the positive sign is divergent and must be excluded. The wave with west-ward propagation cannot exist. There may be an exception. If the equatorial wave guide is bounded by a zonally oriented coast (Gulf of Guinea) the west-ward propagating wave may exist.

In summary we find another equatorial trapped wave,

$$U \sim e^{-\frac{s^2}{2}} \Pi(x - tc). \tag{5.239}$$

It is similar to a coastal Kelvin wave, i.e., non-dispersive and propagates only eastward. Hence, it is called equatorial Kelvin wave.

There is an interesting mathematical representaion of the theory similarly to that of the quantum mechanical harmonic oscillator based on operators. Especially the book of Olbers et al. is recommended for further reading. This mathematical form helps to understand the reflection of equatorial waves at eastern and western continental boundaries. This is the key to understand the mid-lattitude and equatorial wave guide system build from mid-latitude Rossby waves and equatorial waves. It is one of the basic ingredients of the global circulation system.

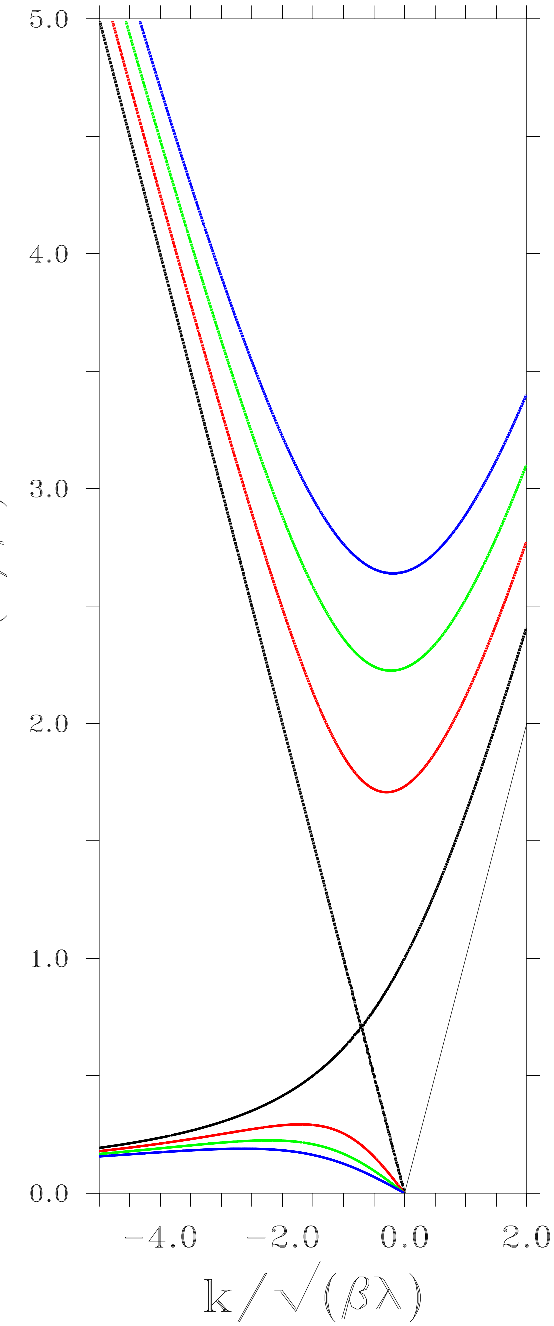


Figure 5.14. The dispersion relation of the equatorial waves spectrum. thin black: $m = -1$, thick black: $m = 0$, red: $m = 1$, green: $m = 2$,

Appendix A

Wind driven currents - details

$$\left.\begin{aligned} u_t + ru - fv = X \\ v_t + rv + fu = Y \end{aligned}\right\} \quad \text{horizontal homogen} \tag{A.0}$$

X, Y: Volumenkraftansatz
rv: Reibungsterm
f: Coriolisparameter

$$u(t=0) = v(t=0) = 0 \qquad u_t t + 2ru_t + \left(r^2 + f^2\right) u = X_t + rX + fY \tag{A.0}$$

Ansatz: Greensche Funktion G(t,t') als Lösung für δ-förmiges Forcing: $G = G(t - t')$
t: Zeitpunkt der Lösung
t': Zeitpunkt des Forcings

$$\begin{aligned} G_t t + 2rG_t + (r^2 - f^2)G = \delta(t - t') \\ RB : G(t, t') = 0 \text{ für } t < t' \text{ (Kausalität)} \end{aligned} \tag{A.0}$$

$\hookrightarrow$ 1. Lösungsansatz:$G_t = -G_{t'} \to \int dt'$
$\hookrightarrow$ 2. Lösungsvariante:

$$\begin{aligned} X_t(t) + rX(t) + fY(t) &= \int_0^\infty \mathrm{d}t' \left(X_t(t') + rX(t') + fY(t')\right) \delta(t - t') \\ \Rightarrow u(\pm) &= \int_0^t dt' \left(X_t(t') + rX(t') + fY(t)\right) G(t - t') \\ &= \int_0^t \mathrm{d}t' F(t') \cdot G(t - t') \end{aligned} \tag{A.0}$$

Wir hatten G bestimmt durch Fouriertransformation:

$$
\begin{aligned}
G(\omega) &= \frac{-1}{2f}\left(\frac{1}{\omega + ir - f} - \frac{1}{\omega + ir + f}\right) \\
G(t-t') &= \frac{\theta(t-t'}{f} e^{-r(t-t')}\sin\left(f(t-t')\right)
\end{aligned}
\tag{A.0}
$$

Beispiel 1: $X(t) = X_0\theta(t)$, $Y(t) = 0$

$$
\begin{aligned}
X_t &= X_0\delta(t) \Rightarrow F(t') = X_0\left(\delta(t') + r\theta(t')\right) \\
u(t) &= X_0\int_0^t \mathrm{d}t'\delta(t')\frac{e^{-r(t-t')}}{f}\sin\left(f(t-t')\right) + X_0 r\int_0^t \mathrm{d}t'\frac{e^{-r(t-t')}}{f}\sin\left(f(t-t')\right) \\
&= \frac{X_0}{f}e^{-rt}\sin\left(ft\right) + \frac{X_0 r}{f}\int_{s=0}^t e^{-rs}\sin\left(fs\right)\mathrm{d}s \\
&= \frac{X_0}{f}e^{-rt}\sin\left(ft\right) + \frac{X_0 r}{f}\int_{s=0}^t \frac{e^{s(if-r)}}{2i}\mathrm{d}s - \frac{X_0 r}{f}\int_{s=0}^t \frac{e^{s(-if-r)}}{2i}\mathrm{d}s \\
&= \frac{X_0}{f}e^{-rt}\sin\left(ft\right) + \left[\frac{X_0 r}{f}\left(\frac{e^{if-r)s}}{-f-ir} - \frac{e^{-if-r)s}}{f-ir}\right)\right]_{s=0}^t \\
&= \frac{X_0}{f}e^{-rt}\sin\left(ft\right) + \frac{X_0 r}{f}\left(-\frac{e^{-rt}}{r^2+f^2}(f\cos\left(ft\right) + r\sin\left(ft\right)) + \frac{f}{r^2+f^2}\right) \\
&= \frac{X_0}{r^2+f^2}\left(fe^{-rt}\sin\left(ft\right) - re^{-rt}\cos\left(ft\right) + r\right)
\end{aligned}
\tag{A.0}
$$

Ohne Beweis:

$$
v(t) = \frac{X_0}{r^2+f^2}\left(fe^{-rt}\cos(ft) + re^{-rt}\sin(ft) - f\right)
\tag{A.0}
$$

Faltungssatz:

$$
f(t) * g(t) = \mathfrak{F}^{-1}\left(\tilde{f}(\omega)\cdot\tilde{g}(\omega)\cdot e^{-i\omega t}\right)
\tag{A.0}
$$

Beispiel:

$$
\begin{aligned}
G(\omega) &= \frac{-1}{2f}\left(\frac{1}{w + ir - f} - \frac{1}{w + ir + f}\right) \\
F(t') &= X_0\left(\delta(t') + r\theta(t')\right) \\
\tilde{F}(\omega) &= X_0 + rX_0\frac{i}{w + i\epsilon}
\end{aligned}
\tag{A.0}
$$

$$
\Rightarrow u(t) = \frac{-X_0}{2f}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}e^{-i\omega t}\left[\frac{1}{w+ir-f} - \frac{1}{w+ir+f} + \frac{ir}{(\omega+ir-f)(\omega+i\epsilon} - \frac{ir}{(\omega+ir+f)(\omega+i\epsilon}\right]
$$

(A.0)

3 Polstellen:

$$\left.\begin{aligned} \omega &= f - ir \\ \omega &= -f - ir \\ \omega &= -i\epsilon \end{aligned}\right\} \qquad \text{alle in unterer Halbebene} \tag{A.0}$$

Also ist erwartungsgem $u(t) = 0$ für $t < 0$. Residuen des Integranden:

$$\begin{aligned}
I(\omega) &= \frac{1}{2\pi} e^{-i\omega t} \left[\frac{1}{w + ir - f} + \frac{-1}{w + ir + f} + \frac{ir}{(\omega + ir - f)(\omega + i\epsilon} + \frac{-ir}{(\omega + ir + f)(\omega + i\epsilon)} \right] \\
\underset{w=f-ir}{Res} I(\omega) &= \frac{e^{-i(f-ir)t}}{2\pi} \left(1 + \frac{ir}{f - ir} \right) = \frac{1}{2\pi i} e^{-rt} e^{-ift} \left(-i + \frac{1r}{-f - ir} \right) \\
\underset{w=-f-ir}{Res} I(\omega) &= \frac{e^{i(f+ir)t}}{2\pi} \left(-1 - \frac{ir}{-f - ir} \right) = \frac{1}{2\pi i} e^{-rt} e^{ift} \left(-i + \frac{1r}{-f - ir} \right) \\
\underset{w=-i\epsilon}{Res} I(\omega) &= \frac{1}{2\pi} \left(\frac{ir}{ir - f} - \frac{ir}{f + ir} \right) = \frac{1}{2\pi i} \frac{2fr}{r^2 + f^2}
\end{aligned} \tag{A.0}$$

Residuensatz:

$$\begin{aligned}
u(t) &= \frac{-X_0}{2f} (-2\pi i) \sum Res \\
&= \frac{X_0}{2f} \left(e^{-rt} e^{-ift} \left(i - \frac{1}{f - ir} \right) + e^{-rt} e^{ift} \left(-i + \frac{1}{-f - ir} \right) + \frac{2f}{r^2 + f^2} \right) \\
&= \frac{X_0}{2f} \left(e^{-rt} 2 \sin(ft) + \frac{2fr}{r^2 + f^2} + \frac{-(-f - ir)e^{-ift} + (f - ir)e^{ift}}{-f^2 - r^2} r e^{-rt} \right) \\
&= \frac{X_0}{r^2 + f^2} \left(f e^{-rt} \sin(ft) - r e^{-rt} \cos(ft) + r \right)
\end{aligned} \tag{A.0}$$

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