Burchard, H., and R. Hofmeister (2008). A dynamic equation for the potential energy anomaly for analysing mixing and stratification in estuaries and coastal seas, Estuarine Coast. Shelf Sci., 77, 679-687.

A detail derivation of equation (14) in Burchard and Hofmeister (2008).

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$$\phi = \frac{1}{D}\int_{-H}^{\eta} gz(\bar{\rho}-\rho)dz \quad (1)$$

$$\frac{\partial \phi}{\partial t} = \frac{g}{D}\frac{\partial}{\partial t}\int_{-H}^{\eta} z(\bar{\rho}-\rho)dz + g\frac{\partial}{\partial t}\left(\frac{1}{D}\right)\int_{-H}^{\eta} z(\bar{\rho}-\rho)dz$$

$$= \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial}{\partial t}(\bar{\rho}-\rho)dz + \frac{g}{D}\eta(\bar{\rho}-\rho_\eta)\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$= \frac{g}{D}\int_{-H}^{\eta} z\left(\frac{\partial \bar{\rho}}{\partial t} - \frac{\partial \rho}{\partial t}\right)dz - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$= \frac{g}{D}\frac{\partial \bar{\rho}}{\partial t}\int_{-H}^{\eta} zdz - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial \rho}{\partial t}dz - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$= \frac{g}{2}\frac{\partial \bar{\rho}}{\partial t}(\eta - H) - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial \rho}{\partial t}dz - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$= \frac{g}{2}(\eta - H)\left[\frac{1}{D}\frac{\partial}{\partial t}\int_{-H}^{\eta}\rho dz + \frac{\partial}{\partial t}\left(\frac{1}{D}\right)\int_{-H}^{\eta}\rho dz\right] - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial \rho}{\partial t}dz - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$= \frac{g}{2D}(\eta - H)\left(\int_{-H}^{\eta}\frac{\partial \rho}{\partial t}dz + \rho_\eta\frac{\partial \eta}{\partial t}\right) - \frac{g}{2D^2}(\eta - H)\frac{\partial D}{\partial t}\int_{-H}^{\eta}\rho dz - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial \rho}{\partial t}dz - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$= \frac{g}{2D}(\eta - H)\int_{-H}^{\eta}\frac{\partial \rho}{\partial t}dz - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial \rho}{\partial t}dz - \frac{g}{2D}(\eta - H)\frac{\partial \eta}{\partial t}\bar{\rho} + \frac{g}{2D}(\eta - H)\rho_\eta\frac{\partial \eta}{\partial t} - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$= \frac{g}{2D}(\eta - H)\int_{-H}^{\eta}\frac{\partial \rho}{\partial t}dz - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial \rho}{\partial t}dz - \frac{g}{2D}(\eta - H)\frac{\partial \eta}{\partial t}\bar{\rho} + \frac{g}{2D}(\eta - H)(\tilde{\rho}_\eta + \bar{\rho})\frac{\partial \eta}{\partial t} - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz$$

$$I.E.\quad \frac{\partial \phi}{\partial t} = \frac{g}{2D}(\eta - H)\int_{-H}^{\eta}\frac{\partial \rho}{\partial t}dz - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial \rho}{\partial t}dz + \frac{g}{2D}(\eta - H)\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} - \frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial \eta}{\partial t} + \frac{g}{D^2}\frac{\partial \eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz \quad (2)$$

$$\frac{\partial \rho}{\partial t} = -u\nabla_h\rho - w\frac{\partial \rho}{\partial z} + \frac{\partial}{\partial z}k_v\frac{\partial \rho}{\partial z} + \nabla_h k_h\nabla_h\rho + Q \quad (3)$$

Substitute the first and second terms of $\frac{\partial \rho}{\partial t}$ in RHS of eq. (2) with eq. (3).

- First, we consider the source term for density, i.e.,

$$\frac{g}{2D}(\eta - H)\int_{-H}^{\eta} Qdz - \frac{g}{D}\int_{-H}^{\eta} zQdz$$

$\eta - H$ is irrelevant with z, we have:

$$\frac{g}{2D}(\eta - H)\int_{-H}^{\eta} Qdz - \frac{g}{D}\int_{-H}^{\eta} zQdz = \frac{g}{D}\int_{-H}^{\eta}\left(\frac{\eta - H}{2} - z\right)Qdz = \frac{g}{D}\int_{-H}^{\eta}\left(\eta - \frac{D}{2} - z\right)Qdz \qquad (4)$$

Now we have got the **term (G)** in equation (14) of Burchard and Hofmeister (2008), hereinafter B & H (2008). Similarly, we can also have the **term (H)** for the horizontal diffusion term $\nabla_h k_h \nabla_h \rho$.

- For the vertical diffusion term, we have:

$$\frac{g}{2D}(\eta - H)\int_{-H}^{\eta}\frac{\partial}{\partial z}k_v\frac{\partial \rho}{\partial z}dz - \frac{g}{D}\int_{-H}^{\eta} z\frac{\partial}{\partial z}k_v\frac{\partial \rho}{\partial z}dz$$

$$= \frac{g}{2D}(\eta - H)\, k_v \frac{\partial \rho}{\partial z}\bigg|_{-H}^{\eta} - \frac{g}{D} z\, k_v \frac{\partial \rho}{\partial z}\bigg|_{-H}^{\eta} + \frac{g}{D}\int_{-H}^{\eta} k_v\frac{\partial \rho}{\partial z}dz$$

$$= \frac{g}{2D}k_v\left(\eta\frac{\partial \rho_\eta}{\partial z} - \eta\frac{\partial \rho_{-H}}{\partial z} - H\frac{\partial \rho_\eta}{\partial z} + H\frac{\partial \rho_{-H}}{\partial z}\right) - \frac{g}{D}k_v\left(\eta\frac{\partial \rho_\eta}{\partial z} + H\frac{\partial \rho_{-H}}{\partial z}\right)$$

$$+ \frac{g}{D}\int_{-H}^{\eta} k_v\frac{\partial \rho}{\partial z}dz$$

$$= -\frac{g}{2D}k_v\left(\eta\frac{\partial \rho_\eta}{\partial z} + \eta\frac{\partial \rho_{-H}}{\partial z} + H\frac{\partial \rho_\eta}{\partial z} + H\frac{\partial \rho_{-H}}{\partial z}\right) + \frac{\rho_0}{D}\int_{-H}^{\eta} P_b dz$$

$$= -\frac{g}{2D}k_v\left(D\frac{\partial \rho_\eta}{\partial z} + D\frac{\partial \rho_{-H}}{\partial z}\right) + \frac{\rho_0}{D}\int_{-H}^{\eta} P_b dz$$

$$= -\frac{\rho_0}{2}\left(P_b^s + P_b^b\right) + \frac{\rho_0}{D}\int_{-H}^{\eta} P_b dz \qquad (5)$$

Now we have got the **terms (E) and (F)** in equation (14) of B & H (2008).

Before further derivation, note that we have following equations:

$$\rho = \tilde{\rho} + \bar{\rho}, \qquad u = \tilde{u} + \bar{u}, \qquad w = \tilde{w} + \bar{w}, \qquad \int_{-H}^{\eta}\tilde{\rho}dz = 0, \int_{-H}^{\eta}\tilde{u}dz = 0$$

$$w\frac{\partial \rho}{\partial z} = (\tilde{w} + \bar{w})\frac{\partial(\tilde{\rho} + \bar{\rho})}{\partial z} = \tilde{w}\frac{\partial \tilde{\rho}}{\partial z} + \bar{w}\frac{\partial \tilde{\rho}}{\partial z} \qquad (6)$$

$$u\nabla_h\rho = \bar{u}\nabla_h\bar{\rho} + \bar{u}\nabla_h\tilde{\rho} + \tilde{u}\nabla_h\bar{\rho} + \tilde{u}\nabla_h\tilde{\rho} \qquad (7)$$

The rest of the first and second terms in RHS of eq. (2) after combining eq. (3) are:

$$-\frac{g}{2D}(\eta - H)\int_{-H}^{\eta}(u\nabla_h\rho + w\frac{\partial \rho}{\partial z})dz + \frac{g}{D}\int_{-H}^{\eta} z(u\nabla_h\rho + w\frac{\partial \rho}{\partial z})dz \quad (8)$$

- Substitute $u\nabla_h\rho$ and $w\frac{\partial \rho}{\partial z}$ in eq. (8) with eqs. (6) and (7), we can have the **terms (C) and (D)** in equation (14) of B & H (2008), then the rest of eq. (8) would be:

$$-\frac{g}{2D}(\eta - H)\int_{-H}^{\eta}(\bar{u}\nabla_h\bar{\rho} + \bar{u}\nabla_h\tilde{\rho} + \tilde{u}\nabla_h\bar{\rho} + \bar{w}\frac{\partial \tilde{\rho}}{\partial z})dz + \frac{g}{D}\int_{-H}^{\eta} z(\bar{u}\nabla_h\bar{\rho} + \bar{u}\nabla_h\tilde{\rho} + \tilde{u}\nabla_h\bar{\rho} + \bar{w}\frac{\partial \tilde{\rho}}{\partial z})dz \quad (9)$$

- First for the horizontal advection term in eq. (9):

$$-\frac{g}{2D}(\eta-H)\int_{-H}^{\eta}(\bar{u}\nabla_h\bar{\rho}+\bar{u}\nabla_h\tilde{\rho}+\tilde{u}\nabla_h\bar{\rho})dz+\frac{g}{D}\int_{-H}^{\eta}z(\bar{u}\nabla_h\bar{\rho}+\bar{u}\nabla_h\tilde{\rho}+\tilde{u}\nabla_h\bar{\rho})dz$$

$$=-\frac{g}{2D}(\eta-H)\left(\bar{u}\nabla_h\bar{\rho}\int_{-H}^{\eta}1dz+\bar{u}\int_{-H}^{\eta}\nabla_h\tilde{\rho}dz+\nabla_h\bar{\rho}\int_{-H}^{\eta}\tilde{u}dz\right)$$

$$+\frac{g}{D}\left(\bar{u}\nabla_h\bar{\rho}\int_{-H}^{\eta}zdz+\bar{u}\int_{-H}^{\eta}z\nabla_h\tilde{\rho}dz+\nabla_h\bar{\rho}\int_{-H}^{\eta}z\tilde{u}dz\right)$$

$$=-\frac{g}{2D}(\eta-H)\bar{u}\int_{-H}^{\eta}\nabla_h\tilde{\rho}dz+\frac{g}{D}\left(\bar{u}\int_{-H}^{\eta}z\nabla_h\tilde{\rho}dz+\nabla_h\bar{\rho}\int_{-H}^{\eta}z\tilde{u}dz\right)$$

$$=-\frac{g}{2D}(\eta-H)\bar{u}\left(\nabla_h\int_{-H}^{\eta}\tilde{\rho}dz-\tilde{\rho}_{\eta}\nabla_h\eta-\tilde{\rho}_{-H}\nabla_hH\right)$$

$$+\frac{g}{D}\left(\bar{u}\int_{-H}^{\eta}z\nabla_h\tilde{\rho}dz+\nabla_h\bar{\rho}\int_{-H}^{\eta}z\tilde{u}dz\right)$$

$$I.E.,\quad -\frac{g}{2D}(\eta-H)\int_{-H}^{\eta}(\bar{u}\nabla_h\bar{\rho}+\bar{u}\nabla_h\tilde{\rho}+\tilde{u}\nabla_h\bar{\rho})dz+\frac{g}{D}\int_{-H}^{\eta}z(\bar{u}\nabla_h\bar{\rho}+\bar{u}\nabla_h\tilde{\rho}+\tilde{u}\nabla_h\bar{\rho})dz$$

$$=\frac{g}{2D}(\eta-H)\bar{u}(\tilde{\rho}_{\eta}\nabla_h\eta+\tilde{\rho}_{-H}\nabla_hH)$$

$$+\underbrace{\frac{g}{D}\bar{u}\int_{-H}^{\eta}z\nabla_h\tilde{\rho}dz}_{(A1)}\qquad\underbrace{+\frac{g}{D}\nabla_h\bar{\rho}\int_{-H}^{\eta}z\tilde{u}dz}_{term\ (B)\ in\ eq.(14)\ of\ B\ \&\ H\ (2008)}\qquad(8)$$

- Second, for the vertical advection term in eq. (9):

$$-\frac{g}{2D}(\eta-H)\int_{-H}^{\eta}\bar{w}\frac{\partial\tilde{\rho}}{\partial z}dz+\frac{g}{D}\int_{-H}^{\eta}z\bar{w}\frac{\partial\tilde{\rho}}{\partial z}dz$$

$$=-\frac{g}{2D}(\eta-H)\left(\int_{-H}^{\eta}\frac{\partial\bar{w}\tilde{\rho}}{\partial z}dz-\int_{-H}^{\eta}\tilde{\rho}\frac{\partial\bar{w}}{\partial z}dz\right)+\frac{g}{D}\int_{-H}^{\eta}z\left(\frac{\partial\bar{w}\tilde{\rho}}{\partial z}-\tilde{\rho}\frac{\partial\bar{w}}{\partial z}\right)dz$$

$$=-\frac{g}{2D}(\eta-H)(\bar{w}\tilde{\rho})|_{-H}^{\eta}+\frac{g}{D}(z\bar{w}\tilde{\rho})|_{-H}^{\eta}-\frac{g}{D}\int_{-H}^{\eta}\bar{w}\tilde{\rho}dz-\frac{g}{D}\int_{-H}^{\eta}z\tilde{\rho}\frac{\partial\bar{w}}{\partial z}dz\qquad(9)$$

Note that in deriving eq. (9) $\int_{-H}^{\eta}\tilde{\rho}\frac{\partial\bar{w}}{\partial z}dz=0$.

Combining $\bar{w}=-\bar{u}\nabla_hH\frac{\eta-z}{D}+\left(\frac{\partial\eta}{\partial t}+\bar{u}\nabla_h\eta\right)\frac{z+H}{D}$ with eq. (9), we have:

$$-\frac{g}{2D}(\eta-H)\int_{-H}^{\eta}\bar{w}\frac{\partial\tilde{\rho}}{\partial z}dz+\frac{g}{D}\int_{-H}^{\eta}z\bar{w}\frac{\partial\tilde{\rho}}{\partial z}dz$$

$$=-\frac{g}{2D}(\eta-H)\left[\left(\frac{\partial\eta}{\partial t}+\bar{u}\nabla_h\eta\right)\tilde{\rho}_{\eta}+\bar{u}\nabla_hH\tilde{\rho}_{-H}\right]+\frac{g}{D}\eta\frac{\partial\eta}{\partial t}\tilde{\rho}_{\eta}+\underbrace{\frac{g}{D}\eta\tilde{\rho}_{\eta}\bar{u}\nabla_h\eta}_{(A2)}$$

$$\underbrace{-\frac{g}{D}H\tilde{\rho}_{-H}\bar{u}\nabla_hH}_{(A3)}-\frac{g}{D}\int_{-H}^{\eta}\bar{w}\tilde{\rho}dz+\underbrace{\frac{g}{D}\int_{-H}^{\eta}z\tilde{\rho}dz\,\nabla_h\bar{u}}_{(A4)}\qquad(10)$$

Finally, combining the horizontal and vertical advection terms (eq.(8) and eq. (10)) and $\frac{g}{2D}(\eta-H)\tilde{\rho}_{\eta}\frac{\partial\eta}{\partial t}-\frac{g}{D}\eta\tilde{\rho}_{\eta}\frac{\partial\eta}{\partial t}+\frac{g}{D^2}\frac{\partial\eta}{\partial t}\int_{-H}^{\eta}z\tilde{\rho}dz$ in eq. (2), we have the following equation:

$$\frac{g}{2D}(\eta-H)\tilde{\rho}_\eta\frac{\partial\eta}{\partial t}-\frac{g}{D}\eta\tilde{\rho}_\eta\frac{\partial\eta}{\partial t}+\frac{g}{D^2}\frac{\partial\eta}{\partial t}\int_{-H}^{\eta}z\tilde{\rho}dz+\frac{g}{2D}(\eta-H)\bar{u}\left(\tilde{\rho}_\eta\nabla_h\eta+\tilde{\rho}_{-H}\nabla_h H\right)$$

$$+\underbrace{\frac{g}{D}\bar{u}\int_{-H}^{\eta}z\nabla_h\tilde{\rho}dz}_{(A1)}\quad\underbrace{+\frac{g}{D}\nabla_h\bar{\rho}\int_{-H}^{\eta}z\tilde{u}dz}_{\textit{term (B) in eq.(14) of B \& H (2008)}}$$

$$-\frac{g}{2D}(\eta-H)\left[\left(\frac{\partial\eta}{\partial t}+\bar{u}\nabla_h\eta\right)\tilde{\rho}_\eta+\bar{u}\nabla_h H\tilde{\rho}_{-H}\right]+\frac{g}{D}\eta\frac{\partial\eta}{\partial t}\tilde{\rho}_\eta+\underbrace{\frac{g}{D}\eta\tilde{\rho}_\eta\bar{u}\nabla_h\eta}_{(A2)}$$

$$-\underbrace{\frac{g}{D}H\tilde{\rho}_{-H}\bar{u}\nabla_h H}_{(A3)}-\frac{g}{D}\int_{-H}^{\eta}\bar{w}\tilde{\rho}dz+\underbrace{\frac{g}{D}\int_{-H}^{\eta}z\tilde{\rho}dz\,\nabla_h\bar{u}}_{(A4)}\qquad(11)$$

And note that:

$$-\frac{g}{D}\int_{-H}^{\eta}\bar{w}\tilde{\rho}dz=\frac{g}{D}\nabla_h\bar{u}\int_{-H}^{\eta}z\tilde{\rho}dz\quad(**)$$

In order to derive the above equation, substitute $\bar{w}$ with $\bar{w}=-\bar{u}\nabla_h H\frac{\eta-z}{D}+\left(\frac{\partial\eta}{\partial t}+\right.$

u∇hηz+HD,

$-\frac{g}{D}\int_{-H}^{\eta}\bar{w}\tilde{\rho}dz=-\frac{g}{D}\int_{-H}^{\eta}\left(\bar{u}\cdot\nabla_h H+\frac{\partial\eta}{\partial t}+\bar{u}\cdot\nabla_h\eta\right)\frac{z}{D}\tilde{\rho}dz=-\frac{g}{D}\int_{-H}^{\eta}(\bar{u}\cdot\nabla_h D-\nabla_h\cdot$

(Du))zDρdz=gD∇hu−Hηzρdz (**), note that *∂η∂t=−∇h·(Du)*.

Then eq. (11) could rewrite as following:

$$\frac{g}{D^2}\frac{\partial\eta}{\partial t}\int_{-H}^{\eta} z\tilde{\rho}dz + \underbrace{\frac{g}{D}\bar{u}\int_{-H}^{\eta} z\nabla_h\tilde{\rho}dz}_{(A1)} \underbrace{+\frac{g}{D}\nabla_h\bar{\rho}\int_{-H}^{\eta} z\tilde{u}dz}_{\textbf{\textit{term}}\ \mathbf{(B)}\ \textbf{\textit{in eq}}.\mathbf{(14)\ of\ B\ \&}\ H\ (2008)} + \underbrace{\frac{g}{D}\eta\tilde{\rho}_\eta\bar{u}\nabla_h\eta}_{(A2)} - \underbrace{\frac{g}{D}H\tilde{\rho}_{-H}\bar{u}\nabla_h H}_{(A3)}$$

$$+\frac{g}{D}\nabla_h\bar{u}\int_{-H}^{\eta} z\tilde{\rho}dz + \underbrace{\frac{g}{D}\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h\bar{u}}_{(A4)}$$

$$= -\frac{g}{D^2}\nabla_h\cdot(D\bar{u})\int_{-H}^{\eta} z\tilde{\rho}dz + \underbrace{\frac{g}{D}\bar{u}\int_{-H}^{\eta} z\nabla_h\tilde{\rho}dz}_{(A1)} + \underbrace{\frac{g}{D}\nabla_h\bar{\rho}\int_{-H}^{\eta} z\tilde{u}dz}_{\textbf{\textit{term}}\ \mathbf{(B)}\ \textbf{\textit{in eq}}.\mathbf{(14)\ of\ B\ \&}\ H\ (2008)}$$

$$+\underbrace{\frac{g}{D}\eta\tilde{\rho}_\eta\bar{u}\nabla_h\eta}_{(A2)} - \underbrace{\frac{g}{D}H\tilde{\rho}_{-H}\bar{u}\nabla_h H}_{(A3)} + \frac{g}{D}\nabla_h\bar{u}\int_{-H}^{\eta} z\tilde{\rho}dz + \underbrace{\frac{g}{D}\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h\bar{u}}_{(A4)}$$

$$= -\frac{g}{D^2}(\bar{u}\cdot\nabla_h D + D\nabla_h\cdot\bar{u})\int_{-H}^{\eta} z\tilde{\rho}dz + \underbrace{\frac{g}{D}\bar{u}\int_{-H}^{\eta} z\nabla_h\tilde{\rho}dz}_{(A1)}$$

$$+ \underbrace{\frac{g}{D}\nabla_h\bar{\rho}\int_{-H}^{\eta} z\tilde{u}dz}_{\textbf{\textit{term}}\ \mathbf{(B)}\ \textbf{\textit{in eq}}.\mathbf{(14)\ of\ B\ \&}\ H\ (2008)} + \underbrace{\frac{g}{D}\eta\tilde{\rho}_\eta\bar{u}\nabla_h\eta}_{(A2)} - \underbrace{\frac{g}{D}H\tilde{\rho}_{-H}\bar{u}\nabla_h H}_{(A3)}$$

$$+\frac{g}{D}\nabla_h\bar{u}\int_{-H}^{\eta} z\tilde{\rho}dz + \underbrace{\frac{g}{D}\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h\bar{u}}_{(A4)}$$

$$= \underbrace{\frac{g}{D}\bar{u}\int_{-H}^{\eta} z\nabla_h\tilde{\rho}dz}_{(A1)} + \underbrace{\frac{g}{D}\nabla_h\bar{\rho}\int_{-H}^{\eta} z\tilde{u}dz}_{\textbf{\textit{term}}\ \mathbf{(B)}\ \textbf{\textit{in eq}}.\mathbf{(14)\ of\ B\ \&}\ H\ (2008)} + \underbrace{\frac{g}{D}\eta\tilde{\rho}_\eta\bar{u}\nabla_h\eta}_{(A2)}$$

$$- \underbrace{\frac{g}{D}H\tilde{\rho}_{-H}\bar{u}\nabla_h H}_{(A3)} + \underbrace{\frac{g}{D}\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h\bar{u}}_{(A4)} \underbrace{-\frac{g}{D^2}\bar{u}\cdot\nabla_h D\int_{-H}^{\eta} z\tilde{\rho}dz}_{(A5)}$$

Combining (A1) to (A5), equal to **term (A)** in equation (14) of B & H (2008), i.e.,

$$\underbrace{\tfrac{g}{D}\bar{u}\textstyle\int_{-H}^{\eta} z\nabla_h\tilde{\rho}dz}_{(A1)} + \underbrace{\tfrac{g}{D}\eta\tilde{\rho}_\eta\bar{u}\nabla_h\eta}_{(A2)} - \underbrace{\tfrac{g}{D}H\tilde{\rho}_{-H}\bar{u}\nabla_h H}_{(A3)} + \underbrace{\tfrac{g}{D}\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h\bar{u}}_{(A4)} \underbrace{-\tfrac{g}{D^2}\bar{u}\cdot\nabla_h D\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz +}_{(A5)}$$

$$\underbrace{\tfrac{g}{D}\nabla_h\bar{\rho}\textstyle\int_{-H}^{\eta} z\tilde{u}dz}_{\textbf{\textit{term}}\ \mathbf{(B)}\ \textbf{\textit{in eq}}.\mathbf{(14)\ of\ B\ \&}\ H\ (2008)}$$

$$= \tfrac{g}{D}\bar{u}\nabla_h\left(\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz\right) + \tfrac{g}{D}\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h\bar{u} - \tfrac{g}{D^2}\bar{u}\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h D + \tfrac{g}{D}\nabla_h\bar{\rho}\textstyle\int_{-H}^{\eta} z\tilde{u}dz$$

$$= \tfrac{g}{D}\bar{u}\nabla_h\left(\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz\right) + \tfrac{g}{D}\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h\bar{u} - \tfrac{g}{D^2}\bar{u}\textstyle\int_{-H}^{\eta} z\tilde{\rho}dz\,\nabla_h D + \tfrac{g}{D}\nabla_h\bar{\rho}\textstyle\int_{-H}^{\eta} z\tilde{u}dz$$

$$= \nabla_h\left(\bar{u}\tfrac{1}{D}\textstyle\int_{-H}^{\eta} gz\tilde{\rho}dz\right) + \tfrac{g}{D}\nabla_h\bar{\rho}\textstyle\int_{-H}^{\eta} z\tilde{u}dz$$

$$= -\nabla_h(\bar{u}\phi) + \tfrac{g}{D}\nabla_h\bar{\rho}\textstyle\int_{-H}^{\eta} z\tilde{u}dz$$

So far, we have all the terms in equation (14) of B & H (2008).

Note that we have not used kinematic boundary conditions given by equation (16) in deriving equation (14).