Hydrodynamics

Lars Umlauf and Hans Burchard

November 29, 2019

Contents

Chapter 1

Basic mathematical tools

When describing physical properties of a fluid, different classes of quantities are appropriate. Some properties, for example temperature, pressure, and energy do not exhibit directional information, and can be fully specified by a single real number. Mathematically, such quantities are described by *scalars*. However, for other variables, for example velocities, forces, and stresses, directional information is essential. These quantities are described by *tensors*. Before examining different ways to work with scalars and tensors, we briefly review the representation of vectors in Cartesian coordinates and provide a mathematical definition of Cartesian tensors.

1.1 Scalar and tensorial quantities

Following the picture of classical mechanics, the kinematics of a moving and deforming fluid is described in the three-dimensional Euclidian space, E_3. The fixed base vectors of the orthonormal base spanning this space are denoted by $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, and the corresponding coordinates are x_1, x_2, and x_3 (or sometimes also x, y, z). The position vector, $\boldsymbol{x}$, of a point in E_3 can be written as

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = \sum_{i=1}^{3} x_i\boldsymbol{e}_i \quad . \tag{1.1}$$

The notation of expressions like (1.1) is considerably simplified by use of the *Einstein summation convention*. According to this convention, the symbol for the sum appearing in (1.1) can be omitted, and the vector $\boldsymbol{x}$ can more compactly be written as

$$\boldsymbol{x} = x_i\boldsymbol{e}_i \quad \text{(summation convention)} \, , \tag{1.2}$$

implying that summation takes place over repeated indices. The summation convention will be used throughout this text.

A property of the space E_3 is that the *dot product* of two (not necessarily orthogonal) unit vectors is defined as the cosine of the angle between these

vectors. It follows that the dot product of two orthogonal unit vectors can be expressed as

$$\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij} \ , \tag{1.3}$$

where δ_{ij} is the *Kronecker delta*. Since the dot product of parallel unit vectors is unity, whereas for orthogonal unit vectors it is zero, the Kronecker delta is defined by

$$\delta_{ij} = \begin{cases} 1 \ , & \text{if } i = j \\ 0 \ , & \text{if } i \neq j \end{cases} \ . \tag{1.4}$$

The components, u_i, of an arbitrary vector, $\boldsymbol{u}$, in the direction of the base vector, $\boldsymbol{e}_i$, can be obtained from the dot product

$$\boldsymbol{u} \cdot \boldsymbol{e}_i = (u_j \boldsymbol{e}_j) \cdot \boldsymbol{e}_i = u_j \delta_{ij} = u_i \ , \tag{1.5}$$

where the summation convention and the properties of the Kronecker delta have been used. The dot product in (1.5) can also be though of as the projection of $\boldsymbol{u}$ in the direction of $\boldsymbol{e}_i$.

1.1.1 Coordinate transformations

As we will see below, the definition of tensors is given in terms of their transformation properties in different coordinates systems. Therefore, it is instructive to consider, in addition to the coordinate system E with base vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, another Cartesian coordinate system, $\overline{E}$, with base vectors $\overline{\boldsymbol{e}}_1$, $\overline{\boldsymbol{e}}_2$, and $\overline{\boldsymbol{e}}_3$. The system $\overline{E}$ is obtained from E by a number of rotations and reflections of axes; an example is shown in Figure 1.1.

As any other vector, the base vectors $\boldsymbol{e}_i$ can be expressed as a linear combination of the base vectors $\overline{\boldsymbol{e}}_j$ according to

$$\boldsymbol{e}_i = a_{ik} \overline{\boldsymbol{e}}_k \ , \tag{1.6}$$

where the components a_{ij} follow from the dot product of (1.6) with $\overline{\boldsymbol{e}}_j$,

$$a_{ij} = \boldsymbol{e}_i \cdot \overline{\boldsymbol{e}}_j \quad . \tag{1.7}$$

The a_{ij} are the *direction cosines*: a_{ij} is the cosine of the angle between the i-axis in the E system and the j-axis in the $\overline{E}$ system. An important property of the direction cosines follows from the dot product of (1.6) with $\boldsymbol{e}_j$,

$$\begin{aligned} \boldsymbol{e}_i \cdot \boldsymbol{e}_j &= a_{ik} \, \overline{\boldsymbol{e}}_k \cdot \boldsymbol{e}_j \\ \Rightarrow \quad \delta_{ij} &= a_{ik} a_{jk} \ , \end{aligned} \tag{1.8}$$

where the last expression in (1.8) illustrates that the a_{ij} are the components of a *unitary matrix*, in which the transpose coincides with the inverse matrix. This is a typical property of pure rotation matrices. Similarly, it can be shown that the inverse transformation law for the base vectors is

$$\overline{\boldsymbol{e}}_i = a_{ki} \boldsymbol{e}_k \ , \tag{1.9}$$

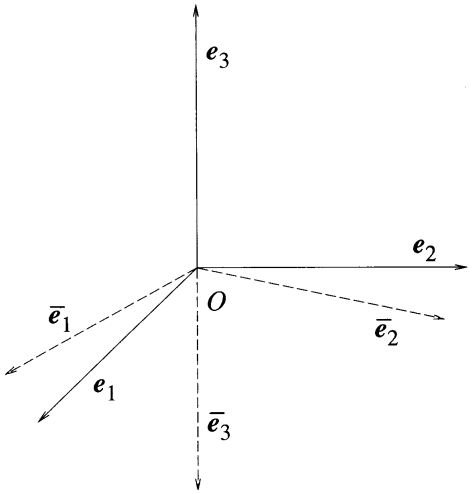

Figure 1.1: A sketch of the E (solid lines) and $\overline{E}$ (dashed lines) coordinate systems. In this particular example, $\overline{E}$ is obtained from E by a reflection of the $\overline{\boldsymbol{e}}_3$ axis, and a rotation in the $\boldsymbol{e}_1$-$\boldsymbol{e}_2$ plane.

and, from scalar multiplication of this equation with $\overline{\boldsymbol{e}}_j$, that

$$\begin{array}{rcl} \overline{\boldsymbol{e}}_i \cdot \overline{\boldsymbol{e}}_j & = & a_{ki}\, \boldsymbol{e}_k \cdot \overline{\boldsymbol{e}}_j \\ \Rightarrow \quad \delta_{ij} & = & a_{ki} a_{kj} \quad . \end{array} \tag{1.10}$$

The position vector, $\boldsymbol{x}$, is the same in both coordinate systems, but its components are different:

$$\boldsymbol{x} = x_i \boldsymbol{e}_i = \overline{x}_j \overline{\boldsymbol{e}}_j \quad . \tag{1.11}$$

The transformation rules for the components are obtained by taking the dot product of (1.11) with $\boldsymbol{e}_k$ and $\overline{\boldsymbol{e}}_k$:

$$\begin{array}{rcl} x_k & = & a_{kj} \overline{x}_j \\ \overline{x}_k & = & a_{jk} x_j \quad . \end{array} \tag{1.12}$$

1.1.2 The definition of Cartesian tensors

We have seen in the context of (1.11) that a key requirement for a vector is that it is identical in all Cartesian coordinate systems. In other words, the length and direction of a vector should not depend on the choice of the coordinate system. Evidently, this concept should also apply for more general directional quantities called *Cartesian tensors*, which we define in the following.

A *zeroth-order tensor* is a scalar. It has $3^0 = 1$ component, which has the same value in all coordinate systems.

A *first-order tensor* is a vector,

$$\boldsymbol{u} = u_i \boldsymbol{e}_i = \overline{u}_j \overline{\boldsymbol{e}}_j \quad . \tag{1.13}$$

It has $3^1 = 3$ components, transforming according to

$$\overline{u}_j = a_{ij} u_i \ , \tag{1.14}$$

and thus insuring that the vector is the same in all Cartesian coordinate systems. A special case of (1.14) is the position vector defined in (1.11).

A *second-order tensor*,

$$\boldsymbol{T} = T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \overline{T}_{kl} \overline{\boldsymbol{e}}_k \otimes \overline{\boldsymbol{e}}_l \ , \tag{1.15}$$

has $3^2 = 9$ components, by definition transforming according to

$$\overline{T}_{ij} = a_{ki} a_{lj} T_{kl} \quad . \tag{1.16}$$

The symbol $\otimes$ introduced in (1.15) represents the *tensor product* and has no equivalent in standard vector notation. Note that some authors simply write $\boldsymbol{e}_i \boldsymbol{e}_j$ to denote the tensor product $\boldsymbol{e}_i \otimes \boldsymbol{e}_j$.

The transformation rule (1.16) guarantees that the tensor $\boldsymbol{T}$ is always the same, irrespective of the particular Cartesian coordinate system used to represent its components. Transformation rules for tensors of arbitrary order can be constructed by a straightforward extension of (1.16). A Cartesian tensor of order N has 3^N components.

1.1.3 Tensor products

In contrast to scalar multiplications, there are a number of different ways to multiply tensorial quantities. The dot product, for example, has already been introduced above. Using the properties of the dot product of two orthonormal base vectors, (1.3), the dot product of two arbitrary vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, can be written as

$$\boldsymbol{u} \cdot \boldsymbol{v} = (u_i \boldsymbol{e}_i) \cdot (v_j \boldsymbol{e}_j) = u_i v_j \delta_{ij} = u_i v_i \quad . \tag{1.17}$$

The so-called *inner product* in E_3 is defined by the implicit relations

$$(\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{w} = (\boldsymbol{v} \cdot \boldsymbol{w}) \boldsymbol{u} \tag{1.18}$$

and

$$\boldsymbol{u} \cdot (\boldsymbol{v} \otimes \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{v}) \boldsymbol{w}. \tag{1.19}$$

This rule can for example be used to evaluate the component form of the inner product of a second-order tensor, $\boldsymbol{A}$, and a vector, $\boldsymbol{b}$, resulting in

$$\begin{aligned} \boldsymbol{A} \cdot \boldsymbol{b} &= (A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot (b_k \boldsymbol{e}_k) = A_{ij} b_k (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \cdot \boldsymbol{e}_k \\ &= A_{ij} b_k (\boldsymbol{e}_j \cdot \boldsymbol{e}_k) \boldsymbol{e}_i = A_{ij} b_k \delta_{jk} \boldsymbol{e}_i \quad . \\ &= A_{ij} b_j \boldsymbol{e}_i \end{aligned} \tag{1.20}$$

Some authors write $\boldsymbol{A}\boldsymbol{b}$ rather than $\boldsymbol{A} \cdot \boldsymbol{b}$ for the product of a tensor and a vector. One can also define the left handed multiplication, $\boldsymbol{b} \cdot \boldsymbol{A}$, which, analogously to the derivation of (1.19), can be shown to correspond to the component form

$$\boldsymbol{b} \cdot \boldsymbol{A} = A_{ij} b_i \boldsymbol{e}_j = A_{ji} b_j \boldsymbol{e}_i \ , \tag{1.21}$$

where in the last step, the dummy indices have been interchanged.

The *cross product* of two (not necessarily orthogonal) unit vectors $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ points into the direction orthogonal to the plane which is spanned by $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ with the magnitude corresponding to the sine of the angle α between the two vectors. Note that the direction depends on the coordinates system (right-handed or left-handed). The cross product of two arbitrary vectors, $\boldsymbol{u}$ and $\boldsymbol{v}$, is given by

$$\boldsymbol{r} = \boldsymbol{u} \times \boldsymbol{v} = u_i v_j \boldsymbol{e}_i \times \boldsymbol{e}_j \ , \tag{1.22}$$

where, because of the orthonormality of the base vectors, $\boldsymbol{e}_i$, the definition of the cross product yields

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \epsilon_{ijk} \boldsymbol{e}_k \ , \tag{1.23}$$

with the so-called *alternating symbol* defined by

$$\varepsilon_{ijk} = \begin{cases} 1 \ , & \text{if } (i,j,k) \text{ cyclic} \\ -1 \ , & \text{if } (i,j,k) \text{ anticyclic} \\ 0 \ , & \text{otherwise} \end{cases} \tag{1.24}$$

Thus, the components of the cross product in (1.22) can compactly be expressed as

$$r_k = \varepsilon_{ijk} u_i v_j \quad . \tag{1.25}$$

Two relations involving the alternating symbol and the Kronecker delta are frequently used. The first is usually called the ε-δ identity,

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl} \ , \tag{1.26}$$

the second,

$$\varepsilon_{ijk} \varepsilon_{ijl} = 2 \delta_{kl} \ , \tag{1.27}$$

has not been assigned a particular name. Both identities can be proven by expanding the corresponding expressions.

1.1.4 Symmetric and skew-symmetric tensors

The *transpose* of second-order tensor, $\boldsymbol{T} = T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$, is defined by

$$\boldsymbol{T}^T = T_{ji} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = T_{ij} \boldsymbol{e}_j \otimes \boldsymbol{e}_i \quad \text{(transpose)} \quad . \tag{1.28}$$

If the tensor $\boldsymbol{T}$ and its transpose coincide, $\boldsymbol{T} = \boldsymbol{T}^T$, the tensor is said to be *symmetric*. In contrast to that, if the relation $\boldsymbol{T} = -\boldsymbol{T}^T$ holds, the tensor is called *anti-symmetric* or *skew-symmetric*. In indical notation, these statements read

$$\begin{array}{rcll} T_{ij} & = & T_{ji} & \text{(symmetric tensor)} \ , \\ T_{ij} & = & -T_{ji} & \text{(skew-symmetric tensor)} \quad . \end{array} \tag{1.29}$$

Evidently, the diagonal elements of any skew-symmetric tensor must be zero, $T_{11} = T_{22} = T_{33} = 0$.

Every second-order tensor can be written as the sum of its symmetric part,

$$T_{(ij)} = \frac{1}{2}(T_{ij} + T_{ji}) \ , \tag{1.30}$$

and skew-symmetric part,

$$T_{[ij]} = \frac{1}{2}(T_{ij} - T_{ji}) \ , \tag{1.31}$$

since the simple relation

$$T_{ij} = T_{(ij)} + T_{[ij]} \tag{1.32}$$

is valid for any second-order tensor.

1.2 Derivatives

1.2.1 Function derivatives

For a one-dimensional, sufficiently smooth function, $f(x)$, the *ordinary derivative* is defined by

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad . \tag{1.33}$$

For a vector-valued function with n arguments $x_1, x_2, \ldots, x_n$, the *partial derivative* with respect to x_i is defined by

$$\frac{\partial}{\partial x_i} f(x_1, \ldots, x_n) = \lim_{\Delta x \to 0} \frac{f(x_1, \ldots, x_i + \Delta x, \ldots, x_n) - f(x_1, \ldots, x_n)}{\Delta x} \quad . \tag{1.34}$$

1.2.2 Derivatives of tensorial quantities

When working with tensorial quantities in E_3, a number of generalised derivatives, playing an important role in fluid mechanics, can be introduced. Generalised derivatives are most conveniently written in symbolic form using the so-called *Nabla-operator*, for an orthonormal basis defined by

$$\nabla = \frac{\partial}{\partial x_i} \boldsymbol{e}_i \ , \tag{1.35}$$

where it should be recalled that the $\boldsymbol{e}_i$ are constant. Using this convention, the *gradient* of a tensorial quantity, $\boldsymbol{\phi}$ is defined by

$$\nabla \boldsymbol{\phi} \equiv \nabla \otimes \boldsymbol{\phi} = \left(\frac{\partial}{\partial x_k} \boldsymbol{e}_k \right) \otimes \boldsymbol{\phi} \quad . \tag{1.36}$$

For example, if $\boldsymbol{\phi} = \boldsymbol{u}(\boldsymbol{x})$ is a vector, then its gradient is the second-order tensor given by

$$\nabla \boldsymbol{u} = \left(\frac{\partial}{\partial x_k} \boldsymbol{e}_k \right) \otimes (u_i \boldsymbol{e}_i) = \frac{\partial u_i}{\partial x_k} \boldsymbol{e}_k \otimes \boldsymbol{e}_i \quad . \tag{1.37}$$

If $\boldsymbol{\phi}$ is a tensor of order N, then its gradient is a tensor of order $N+1$. Similarly, the *divergence* of a tensorial quantity, $\boldsymbol{\phi}$, is defined via the dot product:

$$\nabla \cdot \boldsymbol{\phi} = \left(\frac{\partial}{\partial x_k} \boldsymbol{e}_k \right) \cdot \boldsymbol{\phi} \quad . \tag{1.38}$$

Taking the second-order tensor $\boldsymbol{T}$ as an example, the divergence of $\boldsymbol{T}$ corresponds to the vector

$$\begin{aligned} \nabla \cdot \boldsymbol{T} &= \left(\frac{\partial}{\partial x_k} \boldsymbol{e}_k \right) \cdot (T_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j) = \frac{\partial T_{ij}}{\partial x_k} \delta_{ki} \boldsymbol{e}_j \\ &= \frac{\partial T_{ij}}{\partial x_i} \boldsymbol{e}_j = \frac{\partial T_{ji}}{\partial x_j} \boldsymbol{e}_i \quad . \end{aligned} \tag{1.39}$$

If $\boldsymbol{\phi}$ is a tensor of order N, then its divergence is a tensor of order $N-1$.

The *curl* of a tensorial quantity is evaluated from the cross-product with the Nabla-operator. Evaluated for a vector field, $\boldsymbol{u}(\boldsymbol{x})$, this procedure yields the vector

$$\nabla \times \boldsymbol{u} = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} \boldsymbol{e}_i \quad . \tag{1.40}$$

Finally, the *Laplacian* results from the application of the scalar product of two Nabla-operators, operating on a scalar field, ϕ. The result of this operation is

$$\nabla \cdot \nabla \phi = \nabla^2 \phi = \left(\frac{\partial}{\partial x_i} \boldsymbol{e}_i \right) \cdot \left(\frac{\partial \phi}{\partial x_j} \boldsymbol{e}_j \right) = \frac{\partial}{\partial x_i} \left(\frac{\partial \phi}{\partial x_j} \right) \delta_{ij} = \frac{\partial^2 \phi}{\partial x_i^2} \ , \tag{1.41}$$

thus yielding a scalar expression.

Note that in continuum mechanics and some text on fluid dynamics, a more general way of deriving these gradient expressions is used that is based on the concept of the *directional derivative* (and not on the introduction of the nabla operator) as briefly outlined in the appendix.

1.2.3 Vector identities

Using the above results, a number of identities between tensorial quantities can be derived that will proof useful in later chapters. We start with the example

$$(\nabla \times \boldsymbol{u}) \times \boldsymbol{u} = \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \frac{1}{2} \nabla (\boldsymbol{u} \cdot \boldsymbol{u}) \ , \tag{1.42}$$

which can be derived by introducing the abbreviation $\boldsymbol{a} = \nabla \times \boldsymbol{u}$ in (1.42), and using the definitions for the cross product (1.23) and curl (1.40). This yields

$$\begin{aligned}
(\nabla \times \boldsymbol{u}) \times \boldsymbol{u} &= \boldsymbol{a} \times \boldsymbol{u} \\
&= \varepsilon_{ijk} a_j u_k \boldsymbol{e}_i \\
&= \varepsilon_{ijk} \varepsilon_{jlm} \frac{\partial u_m}{\partial x_l} u_k \boldsymbol{e}_i \\
&= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il}) \frac{\partial u_m}{\partial x_l} u_k \boldsymbol{e}_i \\
&= \left(\frac{\partial u_i}{\partial x_k} u_k - \frac{\partial u_k}{\partial x_i} u_k \right) \boldsymbol{e}_i \\
&= \left(\frac{\partial u_i}{\partial x_k} u_k - \frac{1}{2} \frac{\partial (u_k u_k)}{\partial x_i} \right) \boldsymbol{e}_i \ ,
\end{aligned}$$

where the last line corresponds to the right hand side of (1.42). In row 4, we have made use of the ϵ-δ identity defined in (1.26).

In a similar way it can be shown that

$$\nabla \times (\nabla \cdot \nabla \boldsymbol{u}) = \nabla \cdot \nabla (\nabla \times \boldsymbol{u}) \ , \tag{1.43}$$

implying that the curl of the Laplacian of a vector is identical to the Laplacian of the curl of a vector. Another useful relation is

$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + (\nabla \cdot \boldsymbol{u}) \boldsymbol{\omega} - (\nabla \cdot \boldsymbol{\omega}) \boldsymbol{u} \quad . \tag{1.44}$$

These vector identities will be frequently used in later sections, where it is shown that each of them has a distinct physical interpretation.

Chapter 2

Kinematics

2.1 The description of motion

The motion of a fluid can be thought of as the motion of an infinite number of "fluid particles", moving with the velocity of the fluid. Such a fluid particle is called a *material point* in fluid mechanics. In addition to the definition of the material point, also a *material volume* may be defined as a fluid volume consisting always of the same fluid particles, irrespective of how this volume deforms. In that sense, also a *material surface*, always consisting of the same fluid particles, can be defined. For example, the surface enclosing a material volume is a material surface.

If the motion of all individual material points is known by specifying their position vectors as functions of time, the motion of the whole fluid is known. To identify individual particles of the fluid, we assign to every particle a reference position vector, $\boldsymbol{X}$, at which the particle was located at time $t = t_0$. Then, the position of any particle is given by

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) \; , \tag{2.1}$$

where, according to the definition of the reference vector, $\boldsymbol{X}$, it is required that

$$\boldsymbol{X} = \boldsymbol{\chi}(\boldsymbol{X}, t = t_0) \quad . \tag{2.2}$$

The tip of the position vector, $\boldsymbol{x}$, traces the *path line* of the fluid particle (see Figure 2.1).

Given the motion, (2.1), of the fluid particles, it is straightforward to compute the velocity,

$$\boldsymbol{u} = \frac{\partial \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial t} \; , \tag{2.3}$$

and the acceleration,

$$\boldsymbol{a} = \frac{\partial^2 \boldsymbol{\chi}(\boldsymbol{X}, t)}{\partial t^2} \; , \tag{2.4}$$

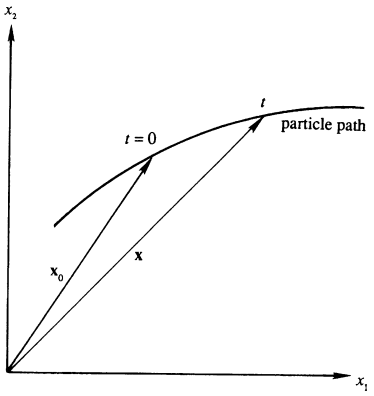

Figure 2.1: Definition of the path line of a fluid particle. Figure taken from Spurk (1989).

of all material points moving with the fluid. The way of specifying the motion of the fluid in terms of a reference position, $\boldsymbol{X}$, and time, t, is called the *Lagrangian* or *material* description.

In contrast to the Lagrangian description, the motion of a fluid can also be specified by prescribing the velocity, $\boldsymbol{u}$, as a function of position in space, $\boldsymbol{x}$, and time, t. This method corresponds to the *Eulerian* or *field* description of the fluid.

Both approaches are closely related. Starting from the Lagrangian description, the Eulerian velocity field can be found by first inverting (2.1) according to

$$\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t) \,, \tag{2.5}$$

and then inserting the result in (2.3). One obtains

$$\boldsymbol{u} = \boldsymbol{u}^L(\boldsymbol{X}, t) = \boldsymbol{u}^L\left(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), t\right) = \boldsymbol{u}^E(\boldsymbol{x}, t) \,, \tag{2.6}$$

which shows that the velocity, $\boldsymbol{u}$, can be specified either as a function, $\boldsymbol{u}^L$, of the Lagrangian variables, $\boldsymbol{X}$ and t, or as a, in general different function, $\boldsymbol{u}^E$, of the Eulerian variables $\boldsymbol{x}$ and t.

On the other hand, if the velocity field, $\boldsymbol{u}^E(\boldsymbol{x}, t)$, is known, the Lagrangian description of a particle's position can be found by solving the three ordinary differential equations

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}^E(\boldsymbol{x}, t) \quad \text{or} \quad \frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i^E(x_j, t) \,, \tag{2.7}$$

where the position of the particle, $\boldsymbol{x}$, is thought to be specified by the function $\boldsymbol{\chi}(\boldsymbol{X}, t)$ according to (2.1). Initial conditions for the position of the particles at $t = t_0$ follow from (2.2). In the following, in accordance with most texts on fluid mechanics, the superscripts "E" and "L" for the Eulerian and Lagrangian description are omitted when their meaning is implicit from the context.

2.2 Streamlines and path lines

As discussed above, the path line of a particle is either determined by the motion, $\boldsymbol{\chi}$, defined in (2.1), or by the set of three differential equations given in (2.7). In a laboratory experiment with a number of fluorescent or reflecting fluid particles, a photograph taken with a very long exposure time would show the path lines of these particles — like a long exposure picture of the starry sky shows the path lines of the stars during the night.

Conducting the same experiment with an "infinitesimally" short exposure time and a large number of fluorescent particles, the motion of each particle would be visible on the photograph as an "infinitesimal" line element, $\mathrm{d}\boldsymbol{x}$, of length, $\mathrm{d}s$, aligned with the direction of the local velocity vector, $\boldsymbol{u}(\boldsymbol{x},t)$. Completing the lines suggested by these line elements would lead to a pattern corresponding to the *streamlines* of the flow. Clearly, if the flow would be unsteady, the streamline pattern would change with time. In general, streamlines and path lines are not the same.

A mathematical definition of the streamlines can be given by considering an arbitrary infinitesimal line element, $\mathrm{d}\boldsymbol{x}$, located at point $\boldsymbol{x}$, and the velocity vector, $\boldsymbol{u}(\boldsymbol{x},t)$. The unit vector in the direction of $\mathrm{d}\boldsymbol{x}$ is given by $\boldsymbol{n}_x = \mathrm{d}\boldsymbol{x}/\mathrm{d}s$. The unit vector in the direction of $\boldsymbol{u}$ is given by $\boldsymbol{n}_u = \boldsymbol{u}/|\boldsymbol{u}|$. However, if $\mathrm{d}\boldsymbol{x}$ is part of a streamline, then, by definition, both unit vectors coincide, $\boldsymbol{n}_x = \boldsymbol{n}_u$. Thus, the streamline may be defined by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} = \frac{\boldsymbol{u}(\boldsymbol{x},t)}{|\boldsymbol{u}|} \quad \text{or} \quad \frac{\mathrm{d}x_i}{\mathrm{d}s} = \frac{u_i(x_j,t)}{(u_k u_k)^{1/2}} \,, \quad (t=\text{const}) \quad . \tag{2.8}$$

Note that, in contrast to the path line defined by (2.7), in (2.8) the time, t, is constant. Therefore, without loss of generality, we can introduce $\mathrm{d}\lambda = \mathrm{d}s/|\boldsymbol{u}|$ in (2.8), and re-write it in the more convenient form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\lambda} = \boldsymbol{u}(\boldsymbol{x},t) \quad \text{or} \quad \frac{\mathrm{d}x_i}{\mathrm{d}\lambda} = u_i(x_j,t) \,, \quad (t=\text{const}) \,, \tag{2.9}$$

where now λ serves as the curve parameter of the streamlines.

Alternatively, it is evident from Figure 2.2 that

$$\frac{\mathrm{d}x}{u(\boldsymbol{x},t)} = \frac{\mathrm{d}y}{v(\boldsymbol{x},t)} = \frac{\mathrm{d}z}{w(\boldsymbol{x},t)} \,, \quad (t=\text{const}) \,, \tag{2.10}$$

leading to three differential equations to determine the streamlines. Which of (2.8), (2.9), or (2.10) is preferable depends on the functional form of the velocity field, $\boldsymbol{u}(\boldsymbol{x},t)$.

In stationary flows, $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x})$, and thus the time dependences of the right hand sides disappear in both, (2.7) and (2.8), defining the path lines and the streamlines, respectively. Under these conditions, one can identify $\mathrm{d}t = \mathrm{d}s/|\boldsymbol{u}|$ in (2.8), which then coincides with (2.7). This implies that in stationary flows, streamlines and path lines are identical, and determined by

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{u}(\boldsymbol{x}) \quad \text{or} \quad \frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i(x_i) \quad . \tag{2.11}$$

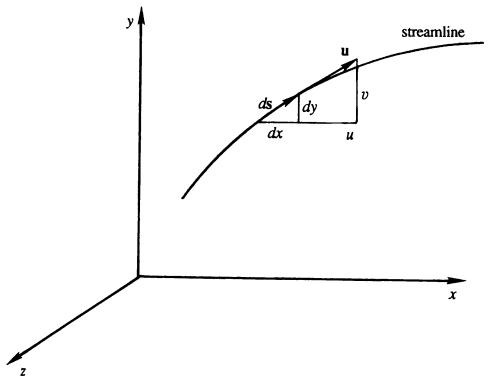

Figure 2.2: Velocity and infinitesimal distances defining the streamline. Figure taken from Spurk (1989).

2.3 The material derivative

In addition to the ordinary and partial derivatives discussed in Section 1.2, the so-called *material derivative* plays a central role in fluid mechanics. To understand this concept, consider the position vector of a material fluid particle, $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$, moving in a specified scalar field, $T(\boldsymbol{x}, t)$. As an example let's imagine that $T(\boldsymbol{x}, t)$ is the temperature of the fluid. The velocity field, according to which the particle is advected, is denoted by $\boldsymbol{u}(\boldsymbol{x}, t)$. An interesting question now is: what temperature change does the particle experience on its path?

We start by defining the *material* or *substantial* derivative,

$$\frac{\mathrm{D}\, T(\boldsymbol{x}, t)}{\mathrm{D}t} \,, \tag{2.12}$$

as the time derivative of the temperature, T, experienced by a particle moving with the fluid, and located at position $\boldsymbol{x}$ at time t. Expressing the position vector, $\boldsymbol{x}$, by the motion defined in (2.1), and using the chain rule for derivatives, we can express the material derivative by

$$\begin{aligned} \frac{\mathrm{D}\, T(\boldsymbol{\chi}(\boldsymbol{X}, t), t)}{\mathrm{D}t} &= \frac{\partial\, T(\boldsymbol{\chi}(\boldsymbol{X}, t), t)}{\partial t} + \frac{\partial \chi_k(\boldsymbol{X}, t)}{\partial t} \frac{\partial\, T(\boldsymbol{\chi}(\boldsymbol{X}, t), t)}{\partial \chi_k(\boldsymbol{X}, t)} \\ &= \frac{\partial\, T(\boldsymbol{x}, t)}{\partial t} + u_k(\boldsymbol{x}, t) \frac{\partial\, T(\boldsymbol{x}, t)}{\partial x_k} \,, \end{aligned} \tag{2.13}$$

where the definition of the velocity, (2.3), has been used. Using (2.1) to substi-

tute for $\boldsymbol{\chi}$, we obtain for the material derivative the expression

$$\begin{aligned} \frac{\mathrm{D}T(\boldsymbol{x},t)}{\mathrm{D}t} &= \frac{\partial T(\boldsymbol{x},t)}{\partial t} + u_k(\boldsymbol{x},t)\frac{\partial T(\boldsymbol{x},t)}{\partial x_k} \quad \text{or} \\ \frac{\mathrm{D}T(\boldsymbol{x},t)}{\mathrm{D}t} &= \frac{\partial T(\boldsymbol{x},t)}{\partial t} + \boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla T(\boldsymbol{x},t)\ , \end{aligned} \tag{2.14}$$

involving only Eulerian fields. The time derivative experienced by a particle moving with a prescribed velocity, $\boldsymbol{w} \neq \boldsymbol{u}$, different from the velocity of the fluid, can be obtained by simply replacing $\boldsymbol{u}$ by $\boldsymbol{w}$ in (2.14). When the particle does not move at all, $\boldsymbol{w} = \boldsymbol{0}$, it is evident from (2.14) that it only feels the *local rate of change*,

$$\frac{\mathrm{D}T(\boldsymbol{x},t)}{\mathrm{D}t} = \frac{\partial T(\boldsymbol{x},t)}{\partial t} \quad . \tag{2.15}$$

Clearly, if the scalar field does not depend on time $T(\boldsymbol{x},t) = T(\boldsymbol{x})$, this local rate of change is zero. Then, a particle moving with the fluid only feels the *advective rate of change*,

$$\frac{\mathrm{D}T(\boldsymbol{x},t)}{\mathrm{D}t} = u(\boldsymbol{x},t)\cdot\nabla T(\boldsymbol{x},t) \quad . \tag{2.16}$$

The concept of the material derivative applies also to tensor fields. For example, the time derivative of the velocity, $\boldsymbol{u}(\boldsymbol{x},t)$, experienced by a particle moving with the fluid, is in component form given by

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial u_i}{\partial t} + u_k\frac{\partial u_i}{\partial x_k}\ , \tag{2.17}$$

or, using symbolical notation,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} \quad . \tag{2.18}$$

Since the time derivative of a fluid particle is its acceleration, $a(\boldsymbol{x},t)$, this equation is the Eulerian equivalent of (2.4).

2.4 Gauss' theorem

Apart from the local concepts of fluid particles or material points, it is often necessary or convenient to work with averaged physical quantities, resulting, for example, from the integral of a physical property over all fluid particles located in a certain volume or on a certain surface inside the fluid. In the following, a few mathematical tools will be introduced which greatly facilitate the manipulation of such quantities.

We start by recalling from elementary calculus that

$$\int_a^b \frac{\mathrm{d}Q(x)}{\mathrm{d}x}\,\mathrm{d}x = Q(b) - Q(a)\ , \tag{2.19}$$

providing a simple relation between an arbitrary scalar, $Q(x)$, evaluated at the fixed boundary points, a and b, and the integral of the gradient of $Q(x)$ over $[a, b]$.

An interesting generalisation of (2.19) for an arbitrary three-dimensional tensor field, $\boldsymbol{Q}(\boldsymbol{x})$, is given by *Gauss' theorem*, stating that

$$\int_V \frac{\partial \boldsymbol{Q}}{\partial x_i} \, \mathrm{d}V = \int_A \boldsymbol{Q} n_i \, \mathrm{d}A \, , \tag{2.20}$$

where V is an arbitrary but simply connected volume, bounded by the surface A. An infinitesimal area element of this surface is denoted by $\mathrm{d}\boldsymbol{A} = \boldsymbol{n}\mathrm{d}A$, where $\boldsymbol{n}$ is the outward unit normal vector located on the surface (see Figure 2.3). Gauss' theorem can be proven for a tensor field, $\boldsymbol{Q}$, of arbitrary order, but the proof is not discussed here.

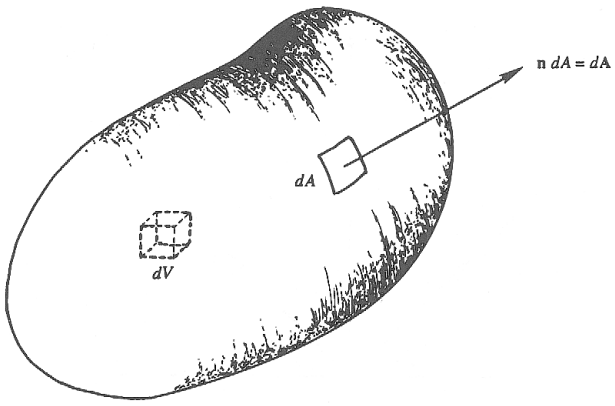

Figure 2.3: Graphical illustration of Gauss' theorem. This figure has been taken from Kundu and Cohen (2008).

For the special case of $\boldsymbol{Q}$ being a vector, Gauss' theorem becomes

$$\int_V \nabla \cdot \boldsymbol{Q} \, \mathrm{d}V = \int_A \boldsymbol{Q} \cdot \boldsymbol{n} \, \mathrm{d}A \quad \text{or} \quad \int_V \frac{\partial Q_i}{\partial x_i} \, \mathrm{d}V = \int_A Q_i n_i \, \mathrm{d}A \, , \tag{2.21}$$

which is, because of the appearance of the divergence, $\nabla \cdot \boldsymbol{Q}$, on the right hand side, usually referred to as the *divergence theorem*. Geometrically, the divergence theorem states that the volume integral over the divergence of a vector equals the surface integral of the outflux of the vector.

2.5 The transport theorems

As we saw above, Gauss' theorem provides a useful tool for the evaluation of a volume integral from knowledge of the boundary values of a physical quantity

alone. Sometimes, however, it is not the volume integral of a physical quantity that is of interest, but the time derivative of the volume integral. If the boundaries of the volume are not fixed in time, as it is in general the case for a material volume, additional complexities arise.

In one spatial dimensional, this problem corresponds to *Leibniz' Rule*,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} Q(x,t)\,\mathrm{d}x &= \int_{a(t)}^{b(t)} \frac{\partial Q(x,t)}{\partial t}\,\mathrm{d}x \\ &+ Q(b(t),t)\frac{\partial b(t)}{\partial t} - Q(a(t),t)\frac{\partial a(t)}{\partial t} \\ &= \int_{a(t)}^{b(t)} \frac{\partial Q(x,t)}{\partial t}\,\mathrm{d}x \\ &+ Q(b(t),t)u_b - Q(a(t),t)u_a\ ,\end{aligned} \tag{2.22}$$

where $Q(x,t)$ is an arbitrary scalar and $a(t)$ and $b(t)$ are the boundary points moving with speeds u_a and u_b, respectively.

Leibniz' Rule can be generalised for an arbitrary tensor field, $\boldsymbol{Q}(\boldsymbol{x},t)$, in E_3 according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \boldsymbol{Q}(\boldsymbol{x},t)\,\mathrm{d}V = \int_V \frac{\partial \boldsymbol{Q}(\boldsymbol{x},t)}{\partial t}\,\mathrm{d}V + \int_A \boldsymbol{Q}\ \boldsymbol{u}_A\cdot\boldsymbol{n}\,\mathrm{d}A\ , \tag{2.23}$$

where $V(t)$ is the moving volume bounded by the moving surface $A(t)$. The vector $\boldsymbol{u}_A$ corresponds to the velocity of the moving boundary. The proof can be found in any standard text book on continuum mechanics. If the velocity, $\boldsymbol{u}_A$, of the bounding surface and the velocity, $\boldsymbol{u}$, of the fluid coincide, the bounding surface moves with the fluid and contains always the same fluid particles. Thus, it is a material surface. Since the volume $V(t)$ is bounded by the surface $A(t)$, it is a material volume. In this case, the generalised Leibniz rule is called *Reynolds' transport theorem*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \boldsymbol{Q}(\boldsymbol{x},t)\,\mathrm{d}V = \int_V \frac{\partial \boldsymbol{Q}(\boldsymbol{x},t)}{\partial t}\,\mathrm{d}V + \int_A \boldsymbol{Q}\ \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}A \quad . \tag{2.24}$$

The surface integral in this equation can be converted into a volume integral by noting that, according to Gauss' theorem, (2.20), one can write in indical notation

$$\int_A (\boldsymbol{Q}\ u_i)n_i\,\mathrm{d}A = \int_V \frac{\partial \boldsymbol{Q}u_i}{\partial x_i}\,\mathrm{d}V = \int_V \left(u_i\frac{\partial \boldsymbol{Q}}{\partial x_i} + \boldsymbol{Q}\frac{\partial u_i}{\partial x_i}\right)\,\mathrm{d}V \quad . \tag{2.25}$$

Introducing this result in (2.24), and making use of the definition of the material derivative, (2.13), an alternative form of Reynolds' transport theorem can be found,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \boldsymbol{Q}(\boldsymbol{x},t)\,\mathrm{d}V = \int_V \left(\frac{\mathrm{D}\boldsymbol{Q}}{\mathrm{D}t} + \boldsymbol{Q}\,\nabla\cdot\boldsymbol{u}\right)\,\mathrm{d}V\ , \tag{2.26}$$

involving only volume integrals.

This last relation can be used to obtain some insight into the meaning of the divergence of the velocity field. Using the special choice $\boldsymbol{Q}(\boldsymbol{x},t)=1$, (2.26) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\mathrm{d}V=\int_{V}\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t)\,\mathrm{d}V \quad . \tag{2.27}$$

Clearly, the left hand side of this integral is the time derivative of the material volume, $\mathrm{d}V(t)/\mathrm{d}t$. The right hand side can be re-written by noting that the mean value theorem of calculus states that, for an arbitrary tensor, $\boldsymbol{B}(\boldsymbol{x})$, we can compute

$$\frac{1}{V}\int_{V}\boldsymbol{B}(\boldsymbol{x})\,\mathrm{d}V=\boldsymbol{B}(\boldsymbol{x}^{*})\;, \tag{2.28}$$

where the point $\boldsymbol{x}^{*}$ is located inside V, such that $\boldsymbol{B}(\boldsymbol{x}^{*})$ is simply the mean value of $\boldsymbol{B}$ evaluated over the volume.

By use of the mean value theorem, (2.27) becomes

$$\frac{1}{V(t)}\frac{\mathrm{d}V(t)}{\mathrm{d}t}=\nabla\cdot\boldsymbol{u}(\boldsymbol{x}^{*},t) \quad . \tag{2.29}$$

Now let the volume become infinitesimally small, $V\to\mathrm{d}V$. Then, $\boldsymbol{x}^{*}\to\boldsymbol{x}$, because $\boldsymbol{x}^{*}$ is always contained in V. Under these conditions, (2.29) becomes

$$\frac{1}{\mathrm{d}V}\frac{\mathrm{D}(\mathrm{d}V)}{\mathrm{D}t}=\nabla\cdot\boldsymbol{u}(\boldsymbol{x},t)\;, \tag{2.30}$$

where the material derivative has been used because we follow the material volume, $\mathrm{d}V$. This expression illustrates that the divergence describes the volume change of a fluid particle moving with the fluid. Flows with $\nabla\cdot\boldsymbol{u}=0$ are called *incompressible*. They play a central role in geophysical fluid mechanics because $\nabla\cdot\boldsymbol{u}=0$ is a very good approximation for many oceanographic and atmospheric flows.

2.6 Rate of deformation and vorticity

We saw in the previous section how the velocity field affects the volume of a material particle. This, however, is only one aspect of the kinematics of a fluid, and many others exist. Having found the relation describing the rate of change of an infinitesimal volume, one might, for example, ask how the direction and length of an infinitesimal line element, $\mathrm{d}\boldsymbol{x}$, travelling with the fluid are affected by the velocity field. This question is discussed in the following.

To introduce the concept of an infinitesimal line element, it is instructive to first look at the deformation of a line element of finite length. The rate of change of this line element, $\boldsymbol{\Delta}_{x}=\boldsymbol{x}_{B}-\boldsymbol{x}_{A}$, connecting two material points located at $\boldsymbol{x}_{A}$ and $\boldsymbol{x}_{B}$, respectively, is given by

$$\dot{\boldsymbol{\Delta}}_{x}=\frac{\mathrm{D}\boldsymbol{x}_{B}}{\mathrm{D}t}-\frac{\mathrm{D}\boldsymbol{x}_{A}}{\mathrm{D}t}=\boldsymbol{u}_{B}-\boldsymbol{u}_{A}=\boldsymbol{\Delta}_{u}\;, \tag{2.31}$$

where $\boldsymbol{u}_A$ and $\boldsymbol{u}_B$ are the instantaneous velocities of the two material points, and $\boldsymbol{\Delta}_u$ their velocity difference.

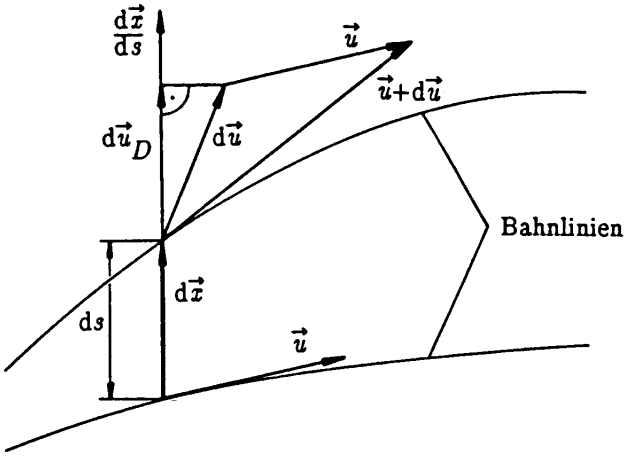

Figure 2.4: Strain acting on an infinitesimal line element. Figure taken from Spurk (1989).

Assuming now that the distance between the two material points becomes infinitesimally small, $\boldsymbol{\Delta}_x$ can be identified with the infinitesimal line element $\mathrm{d}\boldsymbol{x}$ of length $\mathrm{d}s$, and $\boldsymbol{\Delta}_u$ with the infinitesimal velocity difference $\mathrm{d}\boldsymbol{u}$ between the two "endpoints" of $\mathrm{d}\boldsymbol{x}$ (see Figure 2.4). Noting that $\mathrm{d}\boldsymbol{x}$ is a material line element moving with the fluid, the rate term on the left hand side of (2.31) becomes a material derivative, and we have:

$$\frac{\mathrm{D}(\mathrm{d}\boldsymbol{x})}{\mathrm{D}t} = \mathrm{d}\boldsymbol{u} \,, \tag{2.32}$$

indicating that all information about the instantaneous deformation of the infinitesimal line element, $\mathrm{d}\boldsymbol{x}$, is contained in the velocity increment, $\mathrm{d}\boldsymbol{u}$. By definition, the velocity increment, $\mathrm{d}\boldsymbol{u}$, is related to the velocity gradient according to

$$\mathrm{d}u_i = \frac{\partial u_i}{\partial x_j}\mathrm{d}x_j \quad . \tag{2.33}$$

Since, as remarked in the context of (1.30) and (1.31), any second-order tensor can be expressed as the sum of its symmetric and skew-symmetric part, also the velocity gradient may be decomposed according to

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + W_{ij} \quad . \tag{2.34}$$

The symmetric part, $\boldsymbol{S}$, is the *rate of deformation*, defined by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \,, \tag{2.35}$$

and the skew-symmetric part, $\boldsymbol{W}$, is the *vorticity tensor* or *spin tensor*, given by

$$W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \quad . \tag{2.36}$$

Using (2.32), (2.33), and (2.34), the rate of change of an infinitesimal line element can is expressed by

$$\frac{\mathrm{D}(\mathrm{d}x_i)}{\mathrm{D}t} = \mathrm{d}u_i = S_{ij}\mathrm{d}x_j + W_{ij}\mathrm{d}x_j \quad . \tag{2.37}$$

We will see in the following that the two tensors $\boldsymbol{S}$ and $\boldsymbol{W}$ contribute to the fluid motion in very different ways. The first term on the right hand side of (2.37) will be shown to correspond to a distortion of an infinitesimal fluid element, while the second term can be interpreted as a superposed rigid body rotation.

To understand the role of the rate of deformation, $\boldsymbol{S}$, we start by first noting that in mechanics the *rate of strain*, $\mathcal{S}$, of an one-dimensional body of length $l(t)$ (for example an elastic rubber band) is defined as the rate of change of its length, l, with respect to l:

$$\mathcal{S} = \frac{1}{l}\frac{\mathrm{d}l}{\mathrm{d}t} \quad . \tag{2.38}$$

Analogously, the rate of strain of an *infinitesimal* material line element, $\mathrm{d}\boldsymbol{x}$, (see Figure 2.4) can be defined as the rate of change of its length, $\mathrm{d}s$, with respect to $\mathrm{d}s$:

$$\mathcal{S} = \frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} \ , \tag{2.39}$$

where the material derivative has been used because we follow a *material* line element.

Using the relation $\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x}$, (2.39) can be further evaluated according to

$$\begin{aligned}
\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} &= \frac{1}{2}\frac{1}{(\mathrm{d}s)^2}\frac{\mathrm{D}(\mathrm{d}s)^2}{\mathrm{D}t} = \frac{1}{2(\mathrm{d}s)^2}\frac{\mathrm{D}(\mathrm{d}x_i\mathrm{d}x_i)}{\mathrm{D}t} \\
&= \frac{1}{(\mathrm{d}s)^2}\frac{\mathrm{D}(\mathrm{d}x_i)}{\mathrm{D}t}\mathrm{d}x_i = \frac{1}{(\mathrm{d}s)^2}\mathrm{d}u_i\mathrm{d}x_i \\
&= \frac{\partial u_i}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}s}\frac{\mathrm{d}x_i}{\mathrm{d}s} \ ,
\end{aligned} \tag{2.40}$$

where in the last two steps, (2.32) and (2.33) have been used. It is instructive to note that the scalar product, $\mathrm{d}\boldsymbol{u}\cdot\mathrm{d}\boldsymbol{x}$, appearing in the second last term of (2.40), indicates that only the component of $\mathrm{d}\boldsymbol{u}$ in the direction of $\mathrm{d}\boldsymbol{x}$ contributes to straining the line element. This component of the velocity increment has been denoted by $\mathrm{d}\boldsymbol{u}_D$ in Figure 2.4.

Introducing the decomposition of the velocity gradient in symmetric and anti-symmetric parts, $\boldsymbol{S}$ and $\boldsymbol{W}$, into (2.40), and noting that $\boldsymbol{n} = \mathrm{d}\boldsymbol{x}/\mathrm{d}s$ is just the unit vector in the direction of the line element, one obtains

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = (S_{ij} + W_{ij})n_i n_j = S_{ij}n_i n_j \ , \tag{2.41}$$

where the dependence on the skew-symmetric spin tensor, $\boldsymbol{W}$, drops out because $n_i n_j = n_j n_i$ is symmetric.

With (2.41), a clear physical interpretation of the diagonal elements in S_{ij} can be given. To see this, consider, instead of a general orientation of $\mathrm{d}\boldsymbol{x}$, only a line element pointing into the direction of the x_1 axis. This implies $n_1 = 1$, $n_2 = 0$, and $n_3 = 0$ for the components of the unit vector in the element's direction. Then, (2.41) becomes

$$\frac{1}{\mathrm{d}s}\frac{\mathrm{D}(\mathrm{d}s)}{\mathrm{D}t} = S_{11} \ , \tag{2.42}$$

implying that the first diagonal element of the rate of deformation tensor corresponds exactly to the rate of strain of a material element oriented along the x_1-axis. Analogous relations for the other two coordinate directions follow in a similar way.

To analyse the role of the off-diagonal elements in S_{ij}, consider two perpendicular material line elements, $\mathrm{d}\boldsymbol{x}$ and $\mathrm{d}\boldsymbol{x}'$, with lengths $\mathrm{d}s$ and $\mathrm{d}s'$, respectively (see Figure 2.5). Unit vectors in the directions of these elements are denoted by $\boldsymbol{n} = \mathrm{d}\boldsymbol{x}/\mathrm{d}s$ and $\boldsymbol{n}' = \mathrm{d}\boldsymbol{x}'/\mathrm{d}s'$, respectively.

For the magnitude of the component $\mathrm{d}u_R = |\mathrm{d}\boldsymbol{u}_R|$ perpendicular to $\mathrm{d}\boldsymbol{x}$, i.e. parallel to the unit vector $\boldsymbol{n}'$ (see Figure 2.5), one can write

$$\mathrm{d}u_R = \mathrm{d}\boldsymbol{u}\cdot\boldsymbol{n}' = \mathrm{d}\boldsymbol{u}\cdot\frac{\mathrm{d}\boldsymbol{x}'}{\mathrm{d}s'} \quad . \tag{2.43}$$

Similarly, Figure 2.5 illustrates that for the magnitude of the component $\mathrm{d}u'_R = |\mathrm{d}\boldsymbol{u}'_R|$ perpendicular to $\mathrm{d}\boldsymbol{x}'$, i.e. anti-parallel to the unit vector $\boldsymbol{n}$, we obtain

$$\mathrm{d}u'_R = -\mathrm{d}\boldsymbol{u}'\cdot\boldsymbol{n} = -\mathrm{d}\boldsymbol{u}'\cdot\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} \quad . \tag{2.44}$$

Using these results, we can compute the magnitude of the angular velocity of $\mathrm{d}\boldsymbol{x}$, which is simply given by the velocity increment $\mathrm{d}u_R$ divided by the length of the element, $\mathrm{d}s$. According to Figure 2.5, the angular velocity of the mathematically positive angle, φ, may be computed from

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} = -\frac{\mathrm{d}u_R}{\mathrm{d}s} = -\frac{1}{\mathrm{d}s}\left(\mathrm{d}u_i\frac{\mathrm{d}x'_i}{\mathrm{d}s'}\right) = -\frac{\partial u_i}{\partial x_j}\frac{\mathrm{d}x_j}{\mathrm{d}s}\frac{\mathrm{d}x'_i}{\mathrm{d}s'} \ , \tag{2.45}$$

where in the last step, the velocity increment, $\mathrm{d}u_i$ has been expressed with the help of the velocity gradient defined in (2.33). The material derivative in (2.45) indicates that we follow the line elements as they are convected with the flow and rotate. Using very similar geometrical arguments, the angular velocity of the material element $\mathrm{d}\boldsymbol{x}'$ can be shown to be given by

$$\frac{\mathrm{D}\varphi'}{\mathrm{D}t} = -\frac{\mathrm{d}u'_R}{\mathrm{d}s'} = \frac{1}{\mathrm{d}s'}\left(\mathrm{d}u'_i\frac{\mathrm{d}x_i}{\mathrm{d}s}\right) = \frac{\partial u_i}{\partial x_j}\frac{\mathrm{d}x'_j}{\mathrm{d}s'}\frac{\mathrm{d}x_i}{\mathrm{d}s} = \frac{\partial u_j}{\partial x_i}\frac{\mathrm{d}x_j}{\mathrm{d}s}\frac{\mathrm{d}x'_i}{\mathrm{d}s'} \ , \tag{2.46}$$

where in the last step the dummy indices have been interchanged.

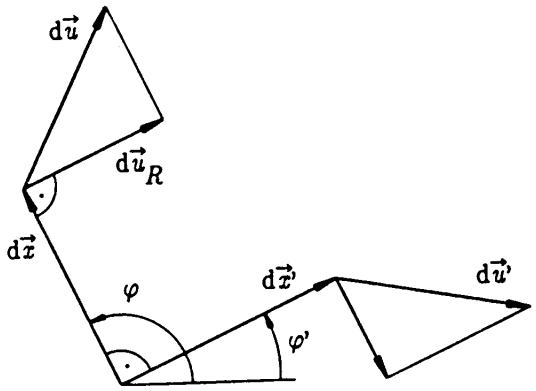

Figure 2.5: Rotation acting on two infinitesimal line elements. Figure from Spurk (1989).

The difference between the rate of change of these two angles is a measure for the *shear rate of strain*, distorting the right angle between the two line elements. Using (2.45) and (2.46), the shear rate of strain becomes

$$\frac{\mathrm{D}\varphi}{\mathrm{D}t} - \frac{\mathrm{D}\varphi'}{\mathrm{D}t} = -\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\frac{\mathrm{d}x_i'}{\mathrm{d}s'}\frac{\mathrm{d}x_j}{\mathrm{d}s} = -2S_{ij}n_i'n_j \quad . \tag{2.47}$$

Assuming, as an example, that the unit vectors of the two line elements point into the coordinate directions x_1 and x_2, i.e. $n_1 = 0$, $n_2 = 1$, and $n_3 = 0$ for $\mathrm{d}\boldsymbol{x}$, and $n_1' = 1$, $n_2' = 0$, and $n_3' = 0$ for $\mathrm{d}\boldsymbol{x}'$, the rate of change of the angle α_{12} between the line elements becomes

$$\frac{\mathrm{D}\alpha_{12}}{\mathrm{D}t} = -2S_{12} \quad . \tag{2.48}$$

This equation clearly illustrates the role of the off-diagonal element S_{12} of the rate of deformation tensor. All other non-diagonal elements of S_{ij} can be interpreted in a similar way.

Similarly to the derivation of (2.47), it is easy to show from (2.45) and (2.46) that the mean value of the rigid body rate of rotation of the two line elements is given

$$\frac{1}{2}\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \frac{\mathrm{D}\varphi'}{\mathrm{D}t}\right) = -W_{ij}n_i'n_j \quad . \tag{2.49}$$

This kinematic relation has a nice interpretation when we introduce the *vorticity* of the velocity field,

$$\omega_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \varepsilon_{ijk}W_{kj} \ , \tag{2.50}$$

which is seen to be the curl of the velocity field. This equation may be inverted,

$$W_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k \ , \tag{2.51}$$

as is readily shown be using the vector identity

$$\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl} \quad . \tag{2.52}$$

Inserting (2.51) into (2.49) yields

$$\frac{1}{2}\left(\frac{\mathrm{D}\varphi}{\mathrm{D}t} + \frac{\mathrm{D}\varphi'}{\mathrm{D}t}\right) = \frac{1}{2}\varepsilon_{ijk}\omega_k n'_i n_j = \frac{1}{2}\,\boldsymbol{\omega}\cdot(\boldsymbol{n}'\times\boldsymbol{n}) \quad . \tag{2.53}$$

If we now consider, as we did in the context of (2.48), two infinitesimal line elements pointing in the x_1 and x_2 directions, (2.53) can be used to compute the rate of change of $\beta_{12} = (\varphi + \varphi')/2$, or the averaged angular velocity,

$$\frac{\mathrm{D}\beta_{12}}{\mathrm{D}t} = \frac{1}{2}\omega_3 \ , \tag{2.54}$$

at which the two elements rotate around the x_3-axis. From this expression it is obvious that the ω_i correspond exactly to *twice* the angular velocity of the averaged rigid body rotation of two line elements oriented along the two coordinate axes perpendicular to the direction x_i.

For $\omega_i = 0$ in the whole domain, the flow is called *irrotational*. Irrotational flows can be greatly simplified by introducing the scalar velocity potential, ϕ, defined by

$$\boldsymbol{u} = \nabla\phi \quad . \tag{2.55}$$

Since the curl of a scalar gradient always vanishes,

$$\nabla\times\nabla\phi = \boldsymbol{0} \ , \tag{2.56}$$

the irrotational character of the flow is seen to be guaranteed by (2.55). Such flows are called *potential* flows. Their great advantage is that, instead of working with three velocity components u_i, one has to work only with one scalar, ϕ.

Chapter 3

Balance laws

The basic principles governing the motion of a fluid correspond to the laws of *classical mechanics* and *thermodynamics*. These laws are axioms. They summarize the experience of physics, and cannot be proven. However, if these axioms are accepted, conclusions can be drawn from them, and very often unimagined results can be obtained.

The basic axioms can be formulated most clearly with respect to an entity called *material body*, consisting always of the same particles or material points. A body is enclosed by a material surface, A, and occupies a material volume V, both of which are variable in time and move according to the kinematic relations discussed in the preceeding chapters.

The concept of fluid particles or material points should not be confused with the concept of molecules[1]. A fluid particle located at point $\boldsymbol{x}$ is thought to represent mean or "expected" values of the physical properties of a large number of molecules located in the neighbourhood of $\boldsymbol{x}$ at time, t. Examples of such mean values are the velocity, $\boldsymbol{u}$, or the mechanical energy, $(\boldsymbol{u}\cdot\boldsymbol{u})/2$. In addition, a fluid particle also represents a number of thermodynamical properties like density, pressure, and temperature that have no equivalent on the molecular level, but sometimes can be computed from molecular properties. The density, ρ, for example, is defined as the average mass of all molecules contained in a volume V, if the volume becomes infinitesimally small,

$$\rho = \lim_{V\to 0} \frac{\Sigma m_i}{V} \ , \tag{3.1}$$

where Σm_i denotes the sum of the molecules' masses. As any other of the averaged fields, the density is assumed to be a *continuous* function of position, $\boldsymbol{x}$, and time, t.

The usefulness of the assumption of continuous fields makes, of course, only sense if there are enough molecules to define stable averages. In particular, if the spatial scales on which the continuous fields change is comparable to the typical

[1] Note that in this text, we consider only one-component bodies consisting of a single type of molecules. No mixtures are discussed.

distance between the molecules, the continuum assumption breaks down. This, however, rarely happens in geophysical applications.

Finally, it should be pointed out that the laws of classical mechanics apply to bodies that are not too small, insuring that quantum effects are negligible, and moving not too fast, such that relativistic effects can be ignored. In this text, we will assume that this is always the case.

3.1 The physics of fluid motion

Using the concept of continuous fields introduced above, a number of essential physical properties of a material body, $\mathcal{B}$, occupying the material volume, V, can be defined. The most important mechanical properties are the *mass*,

$$M = \int_V \rho \, \mathrm{d}V \ , \tag{3.2}$$

the *momentum*,

$$\boldsymbol{P} = \int_V \rho \boldsymbol{u} \, \mathrm{d}V \ , \tag{3.3}$$

the *angular momentum* with reference to the origin,

$$\boldsymbol{A} = \int_V \boldsymbol{x} \times (\rho \, \boldsymbol{u}) \ \mathrm{d}V \ , \tag{3.4}$$

and the *mechanical energy*,

$$K = \int_V \frac{1}{2} \rho \, \boldsymbol{u} \cdot \boldsymbol{u} \, \mathrm{d}V \quad . \tag{3.5}$$

Complementary to the mechanical energy is the thermodynamical concept of *internal energy*,

$$E = \int_V \rho e \, \mathrm{d}V \ , \tag{3.6}$$

where e is the internal energy per unit mass. The internal energy is defined as all energy stored in $\mathcal{B}$ that does not correspond to the mechanical energy, K. There are many ways of changing the internal energy of a body. Examples are the heat exchange of the body's surface with its environment, the deformation of the body, the heat supply to the interior of the body by radiation, and many others.

A thermodynamical property of similar importance as the internal energy is the *entropy*, here denoted as

$$S = \int_V \rho s \, \mathrm{d}V \ , \tag{3.7}$$

where s is the entropy per unit mass. The entropy can be considered as a measure of to what extent the conversion between mechanical and internal energy is reversible. We will come back to this point below.

Having defined the mechanical properties (3.2) - (3.5), we are now in the position of formulating the first principles or axioms of classical mechanics.

1. The first axiom expresses the experience that the mass of a material body does not change with time. This principle is referred to as the *conservation of mass*. For a material body, occupying the material volume V, it can be expressed as
$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0 \,, \tag{3.8}$$
where mass of the body, M, follows from (3.2).

2. *Newton's Second Law* states that the change of momentum of a body, $\mathcal{B}$, equals the sum of the forces, $\boldsymbol{F}$, acting on $\mathcal{B}$,
$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \boldsymbol{F} \,, \tag{3.9}$$
where $\boldsymbol{P}$ has been defined in (3.3). This equation is also called the *balance of momentum*.

3. Independent of Newton's Second Law is the *balance of angular momentum*, stating that the change of angular momentum of a body, $\mathcal{B}$, equals the torque, $\boldsymbol{M}$, of the forces, $\boldsymbol{F}$, acting on $\mathcal{B}$,
$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \boldsymbol{M} \,, \tag{3.10}$$
where $\boldsymbol{A}$ has been defined in (3.4). Is is important to note, however, that the balance of angular momentum is not a first principle of classical physics in the strict sense. There are special types of fluids ("polar fluids") that support internal force couples, which require an extension of (3.10). Nevertheless, for Newtonian fluids, the only type of fluids considered here, (3.10) is valid without restriction.

4. *Newton's Third Law* states that, if a body $\mathcal{B}_1$ exerts a force on second body, $\mathcal{B}_2$, then $\mathcal{B}_2$ exerts the same force on $\mathcal{B}_1$, however, with opposite sign (*actio=reactio*).

5. The *law of gravitation* can be understood by considering a body, $\mathcal{B}_1$, of mass m_1, and a second body, $\mathcal{B}_2$, of mass m_2, connected by a line of length r, parallel to the unit vector, $\boldsymbol{n}$. Then, the gravity force, $\boldsymbol{G}$, between the two bodies given by
$$\boldsymbol{G} = \gamma \frac{m_1 m_2}{r^2}\, \boldsymbol{n} \,, \tag{3.11}$$
where γ is the universal constant of gravity.

These principles constitute the foundations of classical mechanics. With the exception of the first, they are due to the British physicist and astromer Sir Isaak Newton (1643-1727).

In addition to the mechanical principles, two thermodynamical laws concerning the internal energy and entropy are of essential importance for fluid mechanics.

1. According to the *balance of energy*, the change of mechanical plus the change of internal energy of a body, $\mathcal{B}$, is equal to the rate of work done by the mechanical forces acting on the body, the internal heat supply, R, and the total heat flux, Q, through the boundary of $\mathcal{B}$, The mechanical forces are the rate of work, P_V, of the external forces acting in the interior of the body (e.g. gravity and magnetic forces), and the rate of work, P_A, of the forces acting on the body's surface (surface stresses). Then, the balance of energy can be written

$$\frac{\mathrm{d}(K+E)}{\mathrm{d}t} = P_V + P_A + R + Q \;, \tag{3.12}$$

   where K and E are given by (3.5) and (3.6), respectively. This relation is also known as the *first law of thermodynamics*.

2. The *second law of thermodynamics* is an axiom about the reversibility of thermodynamical processes, like the observation that heat always flows from high to low temperature or that the mechanical energy can be converted into heat via friction, but not the other way around.

   The application of the second law requires that we postulate the existence of three variables, defined for any body, $\mathcal{B}$. The first is a variable called entropy, S, the second a variable called entropy flux, Q_S, through the boundary of $\mathcal{B}$, and third a variable called entropy supply, R_S, in the interior of $\mathcal{B}$. Then, the second law states that the total rate of change of the entropy can never be smaller as the sum of the entropy flux and the entropy supply,

$$\frac{\mathrm{d}S}{\mathrm{d}t} \geq Q_S + R_S \;, \tag{3.13}$$

   which implies that, if there is no flux and supply of entropy for $\mathcal{B}$, the entropy of $\mathcal{B}$ cannot decrease. This statement alone is not very useful, unless a few further postulates are made. The general agreement about these additional postulates is limited, and we refer here to the most traditional view.

   This view requires that we postulate the existence of an always positive *absolute temperature*, T. Additionally, we postulate that the entropy flux, Q_S, is the heat flux divided by the absolute temperature, and that the entropy supply, R_S, is the heat supply divided by the absolute temperature. The special form of (3.13) obtained this way is called the *Clausius-Duhem* inequality. These postulates are not at all obvious. However, as we will see below they lead to a number of valuable physical results (e.g. to the above mentioned results that heat flows from high to low temperature) that cannot be obtained in any other way.

In addition to these laws, which are valid for an arbitrary body, $\mathcal{B}$, a specification is needed to describe the material of the body, e.g. whether it is rigid, plastic, elastic, etc. — or a fluid. The *theory of materials* is occupied with the

consistent specification of such *material laws*. Before we discuss these laws, however, we will investigate in the following the consequences of the fundamental principles of mechanics and thermodynamics.

3.2 Balance of mass

Identifying $\boldsymbol{Q} = \rho$ in (2.26), the conservation of mass can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho \,\mathrm{d}V = \int_V \left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\frac{\partial u_i}{\partial x_i}\right) \mathrm{d}V = 0 \quad . \tag{3.14}$$

This equation is valid for an arbitrary material volume. Obviously, for a special choice of the volume, the integral could vanish even for a non-zero integrand. This, however, is not so if the volume is arbitrary. For arbitrary volumes and continuous fields, the only way to insure that (3.14) is satisfied for all integrands, is to let the integrand vanish. This leads to the local form of the conservation of mass,

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\frac{\partial u_i}{\partial x_i} = 0 \quad . \tag{3.15}$$

A special case of (3.15) are flows with

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \,, \tag{3.16}$$

indicating that the density of a material particle does not change with time. According to (3.15), such flows also have zero divergence,

$$\frac{\partial u_i}{\partial x_i} = 0 \,, \tag{3.17}$$

and thus preserve the volume of material particles as shown in (2.30). Flows obeying (3.17) are therefore called *incompressible* flows. Note that (3.16) does not imply that the density is the same in the whole fluid. This is only the case for flows with *homogeneous* density, obeying

$$\frac{\partial \rho}{\partial x_i} = 0 \quad . \tag{3.18}$$

Using Reynolds' transport theorem, (2.24), it is easy to show that (3.14) corresponds to the integral statement

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho \,\mathrm{d}V = \int_V \frac{\partial \rho}{\partial t} \,\mathrm{d}V + \int_A \rho \, u_i n_i \,\mathrm{d}A = 0 \quad . \tag{3.19}$$

Considering a fixed volume, coinciding with the moving volume, $V(t)$, at time t, this form of the conservation of mass has a simple interpretation: the integrated

change of mass inside the volume is equal to the averaged flux of mass through its boundary. If the density field is stationary,

$$\frac{\partial \rho}{\partial t} = 0 \,, \tag{3.20}$$

the balance of mass, (3.19), reduces to

$$\int_A \rho \, u_i n_i \, \mathrm{d}A = 0 \,, \tag{3.21}$$

which simply indicates that the average flow of mass across the boundary of a body is zero. For an illustration, see Figure 3.1.

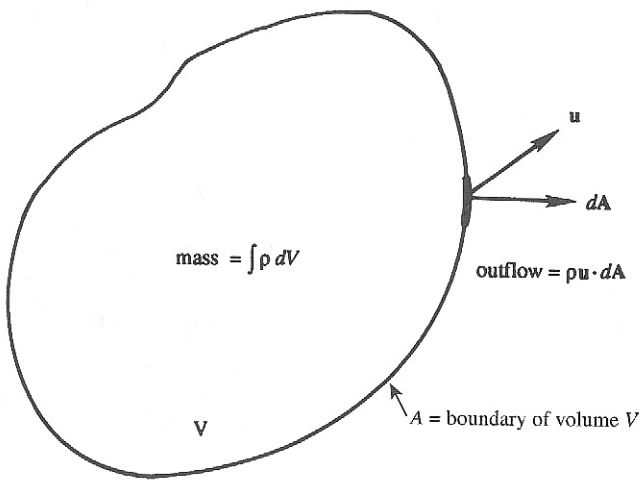

Figure 3.1: Graphical illustration of the mass conservation of a volume fixed in space. This figure has been taken from Kundu and Cohen (2008).

3.3 Balance of momentum

Newton' Second Law states that the change of momentum equals the sum of the forces acting on the body. These forces can be distinguished between body forces, like gravity or magnetic forces, acting on each material particle inside the body, and surface forces, acting only on particles located at the body's surface. The surface force acting on an infinitesimal surface element, $\boldsymbol{n}\mathrm{d}A$, divided by the area, $\mathrm{d}A$, of the surface element is referred to the stress vector, $\boldsymbol{t}$. Using this definition, the balance of momentum, (3.9), can be re-written

$$\frac{\mathrm{d}P_i}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho u_i \, \mathrm{d}V = \int_V \rho k_i \, \mathrm{d}V + \int_A t_i \, \mathrm{d}A \,, \tag{3.22}$$

where the vector of the body force has been denoted by $\boldsymbol{k}$. This vector has the units of a force per mass. Note that the quantity $\rho\boldsymbol{k}$ has the units of a force per volume, such that $\rho\boldsymbol{k}\mathrm{d}V$ is the infinitesimal force acting inside the volume element, $\mathrm{d}V$.

The left hand side of (3.22) can be re-expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_i \,\mathrm{d}V = \int_V \rho \frac{\mathrm{D}u_i}{\mathrm{D}t}\,\mathrm{d}V \quad . \tag{3.23}$$

To understand this relation, we apply (2.26) with $\boldsymbol{Q} = \rho\boldsymbol{B}$, where $\boldsymbol{B}$ is an arbitrary tensor, leading to

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\boldsymbol{B}\,\mathrm{d}V &= \int_V \left(\frac{\mathrm{D}(\rho\boldsymbol{B})}{\mathrm{D}t} + \rho\boldsymbol{B}\frac{\partial u_i}{\partial x_i}\right)\mathrm{d}V \\
&= \int_V \left[\rho\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t} + \boldsymbol{B}\left(\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\,\frac{\partial u_i}{\partial x_i}\right)\right]\mathrm{d}V \\
&= \int_V \rho\frac{\mathrm{D}\boldsymbol{B}}{\mathrm{D}t}\,\mathrm{d}V\ ,
\end{aligned} \tag{3.24}$$

where for the last step, the balance of mass, (3.15), has been invoked. For the special case $\boldsymbol{B} = \boldsymbol{u}$, (3.23) follows, and (3.22) reads

$$\int_V \rho\left(\frac{\mathrm{D}u_i}{\mathrm{D}t} - k_i\right)\mathrm{d}V = \int_A t_i\,\mathrm{d}A \quad . \tag{3.25}$$

Clearly, if it would be possible to re-write the integrand on the right hand side as the product of some quantity times the unit normal vector, $\boldsymbol{n}$, we could convert the surface integral into a volume integral with the help of Gauss' theorem. Then, by use of the same arguments leading to the local conservation of mass, (3.15), we would be able to derive a local statement of the balance of momentum.

3.3.1 The stress tensor

Every deformed material body, whether it consists of a fluid, an elastic or any other material, exhibits an internal *state of stress*, which is, in general, a function of position, $\boldsymbol{x}$, and time, t. An interesting question is what quantity completely describes this state of stress. The stress vector, $\boldsymbol{t}$, the most obvious candidate for this quantity, cannot contain all information about the state of stress at a point, $\boldsymbol{x}$, as can be seen by the following argument. If we "cut" the material body along a plane containing $\boldsymbol{x}$, the stress vector at $\boldsymbol{x}$ becomes "visible". It can be imagined as the stress "felt" by a thin layer of glue used to hold both parts of the material body together. This stress, however, depends in general on the orientation of the cutting plane, defined by the unit outward normal vector, $\boldsymbol{n}$. Thus, the general dependence of the stress vector is $\boldsymbol{t}(\boldsymbol{x}, t; \boldsymbol{n})$.

To investigate the particular functional form in which $\boldsymbol{t}$ depends on $\boldsymbol{n}$, consider the tetrahedron depicted in Figure 3.2. Imagine that this tetrahedron is

located inside a deforming material body. Then, the tetraheron itself can be considered as a material body, and (3.25) can be applied. Assuming that V is the volume of the tetraherdron, (3.25) becomes

$$\begin{aligned}\int_V \rho\left(\frac{\mathrm{D}u_i}{\mathrm{D}t} - k_i\right)\,\mathrm{d}V &= \int_{A_1} t_i(-\boldsymbol{e}_1)\,\mathrm{d}A_1 + \int_{A_2} t_i(-\boldsymbol{e}_2)\,\mathrm{d}A_2 \\ &+ \int_{A_3} t_i(-\boldsymbol{e}_3)\,\mathrm{d}A_3 + \int_{A_n} t_i(\boldsymbol{n})\,\mathrm{d}A_n\ ,\end{aligned} \tag{3.26}$$

where the A_i denote the tetrahedron's surfaces with outward unit normal vectors $-\boldsymbol{e}_i$, and A_n the tilted surface with normal vector $\boldsymbol{n}$, see Figure 3.2. Note, that the dependence of $\boldsymbol{t}$ on the arguments $\boldsymbol{x}$ and $\boldsymbol{t}$ has been supressed here and in the following for compactness. Acccording to the mean value theorem, the

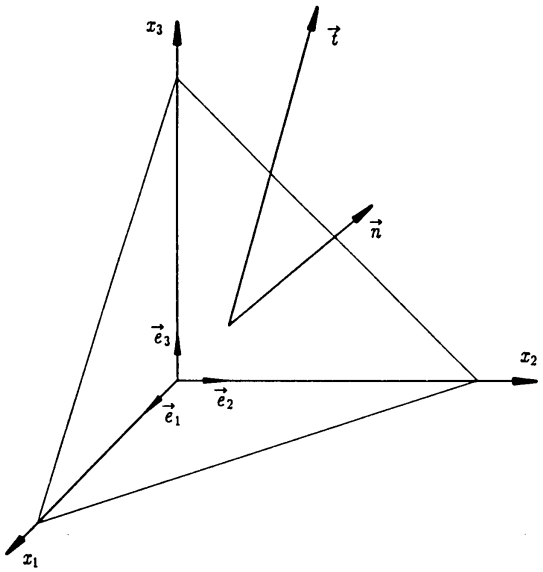

Figure 3.2: Stresses acting on a tetrahedron. Figure from Spurk (1989).

volume integral on the left hand side of (3.26) can be expressed as the product of the volume, V, of the tetrahedron and the value of the integrand evaluated at some point, $\boldsymbol{x}^*$, inside the volume. Likewise, each of the surface integrals can be expressed as the product of the area of the surface, respectively, and the integrand evaluated at some point on the surface. Thus, (3.26) becomes

$$\begin{aligned}V\left[\rho\left(\frac{\mathrm{D}u_i}{\mathrm{D}t} - k_i\right)\right]_{x=x^*} &= A_1\left[t_i(-\boldsymbol{e}_1)\right]_{\boldsymbol{x}=\boldsymbol{x}^{(1)}} + A_2\left[t_i(-\boldsymbol{e}_2)\right]_{\boldsymbol{x}=\boldsymbol{x}^{(2)}} \\ &+ A_3\left[t_i(-\boldsymbol{e}_3)\right]_{\boldsymbol{x}=\boldsymbol{x}^{(3)}} + A_n\left[t_i(\boldsymbol{n})\right]_{x=x^{(n)}}\ ,\end{aligned} \tag{3.27}$$

where the points $\boldsymbol{x}^{(i)}$ are located somewhere on the surfaces A_i, and the point $\boldsymbol{x}^{(\boldsymbol{n})}$ is located somewhere on the tilted surface, A_n.

This equation can be further simplified by noting that the area of the surfaces, A_i, is related to the tilted surface, A_n, according to the simple relation

$$A_i = n_i A_n \ , \tag{3.28}$$

which follows from the geometry of a tetrahedron. Additionally, Newton's Third Law, *actio=reactio*, requires that

$$t_i(-\boldsymbol{e}_i) = -t_i(\boldsymbol{e}_i) \ , \tag{3.29}$$

which states that the forces exerted by the material inside the tetraherdon on the material outside are simply the negative of the forces exerted by the material outside the tetraherdron on the material inside. Relation (3.29) is sometimes called the *Cauchy Lemma*.

Using (3.28) and (3.29), the momentum balance (3.27) can be re-written as

$$\begin{aligned} \frac{V}{A_n}\left[\rho\left(\frac{\mathrm{D}u_i}{\mathrm{D}t} - k_i\right)\right]_{x=x^*} \quad = \quad & -n_1\left[t_i(\boldsymbol{e}_1)\right]_{\boldsymbol{x}=\boldsymbol{x}^{(1)}} \\ & -n_2\left[t_i(\boldsymbol{e}_2)\right]_{\boldsymbol{x}=\boldsymbol{x}^{(2)}} \\ & -n_3\left[t_i(\boldsymbol{e}_3)\right]_{\boldsymbol{x}=\boldsymbol{x}^{(3)}} \\ & +\left[t_i(\boldsymbol{n})\right]_{x=x^{(\boldsymbol{n})}} \quad . \end{aligned} \tag{3.30}$$

If l is a typical length of the tetrahedron, for example the length of one of the edges, the volume of the tetrahedron is proportial to the cube of this length, $V \propto l^3$. Similarly, for the area of the tilted surface one finds $A_n \propto l^2$. If we now let the size of the tetrahedron shrink to zero, $l \to 0$, the left hand side of (3.30) is proportional to l, and becomes negligible compared to the right hand side. Also, because of the meanvalue theorem, for $l \to 0$, we have $\boldsymbol{x}^{(i)} \to \boldsymbol{x}$ and $\boldsymbol{x}^{(n)} \to \boldsymbol{x}$. Thus, (3.30) adopts the simple form

$$-t_i(\boldsymbol{e}_1)n_1 - t_i(\boldsymbol{e}_2)n_2 - t_i(\boldsymbol{e}_3)n_3 + t_i(\boldsymbol{n}) = 0 \ , \tag{3.31}$$

where now all terms are evaluated at the same point, $\boldsymbol{x}$. Using the summation convention this equation becomes

$$t_i(\boldsymbol{n}) = t_i(\boldsymbol{e}_j)n_j \quad . \tag{3.32}$$

This last equation implies that if we know the stress vector on all three coordinate planes of the tetrahedron, then the stress vector on the tilted surface is known. Introducing the abbreviation

$$T_{ji} = t_i(\boldsymbol{e}_j) \ , \tag{3.33}$$

relation (3.32) can be written as

$$t_i(\boldsymbol{n}) = T_{ji}n_j \tag{3.34}$$

where T_{ij} are the components of a tensor called the *Cauchy stress tensor*, $\boldsymbol{T}$. This tensor, $\boldsymbol{T}$, contains all information to compute the stress vector, $\boldsymbol{t}$, on arbitrary surfaces inside the body and thus uniquely determines that state of stress in a material. It follows from the definition (3.33) that T_{ij} is the component of the stress vector in the direction $\boldsymbol{e}_j$, measured on the plane with the outward unit normal vector $\boldsymbol{e}_i$ (i.e. a coordinate plane). This is illustrated in Figure 3.3.

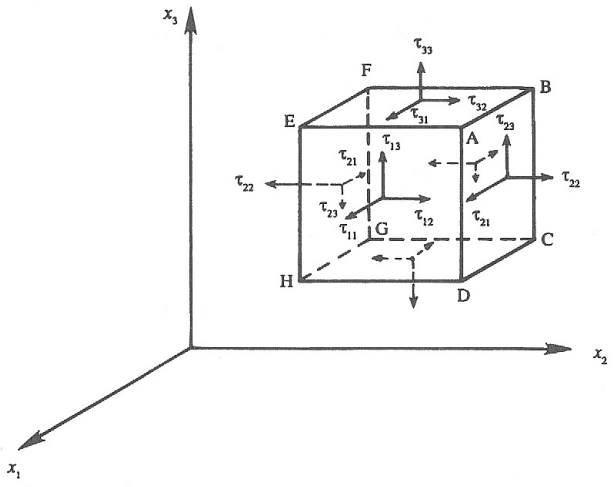

Figure 3.3: Graphical representation of a stress at a point. This figure has been taken from Kundu and Cohen (2008).

The crucial result in (3.34) indicates that the stress vector, $\boldsymbol{t}$, is a linear function of the unit normal vector, $\boldsymbol{n}$. We showed this here by using a classical proof of continuum mechanics called *Cauchy's tetrahedron argument.*

3.3.2 Integral and local balance of momentum

Using the relation between $\boldsymbol{t}$ and $\boldsymbol{T}$, (3.34), and Gauss' theorem, (2.20), the last term in (3.22) can be transformed into a volume integral. Thus, (3.22) can be written as

$$\int_V \left(\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} - \rho k_i - \frac{\partial T_{ji}}{\partial x_j} \right) \mathrm{d}V = 0 \quad . \tag{3.35}$$

Again, the integration volume is arbitrary, and the local statement of the balance of momentum can be derived,

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = \rho k_i + \frac{\partial T_{ji}}{\partial x_j} \quad . \tag{3.36}$$

Another useful integral statement of the balance of momentum can be derived from (3.22). Therefore, we first note that with the help of Reynolds'

transport theorem, (2.24), the left hand side (3.22) can be re-written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_i \,\mathrm{d}V = \int_V \frac{\partial(\rho u_i)}{\partial t}\,\mathrm{d}V + \int_A \rho u_i (u_j n_j)\,\mathrm{d}A \quad . \tag{3.37}$$

Next, we assume that the volume force has a potential, $\rho\boldsymbol{k} = -\nabla\Phi$. For constant density, ρ, this would for example be the case for $\boldsymbol{k}$ corresponding to the gravity force. Using this potential, the first term on the right hand side of (3.22) can be expressed as

$$\int_V \rho k_i \,\mathrm{d}V = -\int_V \frac{\partial\Phi}{\partial x_i}\,\mathrm{d}V = -\int_A \Phi n_i \,\mathrm{d}A \;, \tag{3.38}$$

where Gauss' theorem, (2.20), has been used. Thus, with the help of (3.37) and (3.38), the balance of momentum, (3.22), becomes

$$\int_V \frac{\partial(\rho u_i)}{\partial t}\,\mathrm{d}V = -\int_A \rho u_i (u_j n_j)\,\mathrm{d}A - \int_A \Phi n_i \,\mathrm{d}A + \int_A t_i \,\mathrm{d}A \quad . \tag{3.39}$$

The importance of this equation becomes apparent by noting that for stationary flows, the balance of momentum can be evaluated from physical and geometrical quantities at the boundaries only.

3.4 Balance of angular momentum

It has been mentioned above that the balance of angular momentum, (3.10), is frequently used in fluid mechanics although it cannot be considered a first principle of classical physics in the strict since (like Newton's Laws). The reason for this is that some special types of fluids (polar fluids), which include liquid crystals and ferromagnetic fluids, can support internal force couples that need to be taken into account when formulating a complete balance of angular momentum. Here, we restrict ourselves exclusively to non-polar fluids, for which (3.10) is valid without restriction.

Under this condition, the same body and surface forces responsible for the change of momentum in (3.22) also lead to a torque, changing the angular momentum at a rate

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \boldsymbol{x}\times(\rho\boldsymbol{u})\,\mathrm{d}V = \int_V \boldsymbol{x}\times(\rho\boldsymbol{k})\,\mathrm{d}V + \int_A \boldsymbol{x}\times\boldsymbol{t}\mathrm{d}A \;, \tag{3.40}$$

or, in indical notation,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \varepsilon_{ijk} x_j \rho u_k \,\mathrm{d}V = \int_V \varepsilon_{ijk} x_j \rho k_k \,\mathrm{d}V + \int_A \varepsilon_{ijk} x_j t_k \mathrm{d}A \quad . \tag{3.41}$$

In this particular case, we took the origin as the reference point to define the angular momentum. We will see in the following, however, the the choice of the reference point is completely irrelevant.

The left hand side of (3.41) can be re-written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \varepsilon_{ijk} x_j \rho u_k \,\mathrm{d}V = \int_V \varepsilon_{ijk}\rho \frac{\mathrm{D}(x_j u_k)}{\mathrm{D}t}\,\mathrm{d}V \;, \tag{3.42}$$

if we identify the components of the arbitrary tensor $\boldsymbol{B}$ with $B_{jk} = x_j u_k$ in (3.24).

Using (3.34) for the stress vector, and Gauss' Theorem, (2.20), the surface integral on the right hand side of (3.41) becomes

$$\int_A \varepsilon_{ijk} x_j t_k \,\mathrm{d}A = \int_A \varepsilon_{ijk} x_j T_{lk} n_l \,\mathrm{d}A = \int_V \varepsilon_{ijk} \frac{\partial (x_j T_{lk})}{\partial x_l}\,\mathrm{d}V \quad . \tag{3.43}$$

With the help of (3.42) and (3.43), the balance of angular momentum, (3.41), can be expressed in terms of volume integrals only,

$$\int_V \varepsilon_{ijk}\left(\rho \frac{\mathrm{D}(x_j u_k)}{\mathrm{D}t} - \frac{\partial (x_j T_{lk})}{\partial x_l} - x_j \rho k_k\right)\mathrm{d}V = 0 \;, \tag{3.44}$$

which becomes

$$\int_V \left[\varepsilon_{ijk} x_j \left(\rho \frac{\mathrm{D}u_k}{\mathrm{D}t} - \rho k_k - \frac{\partial T_{lk}}{\partial x_l}\right) + \rho \varepsilon_{ijk} u_j u_k - \varepsilon_{ijk} T_{jk}\right]\mathrm{d}V = 0 \;, \tag{3.45}$$

after carying out the derivatives and noting that $u_i = \mathrm{D}x_i/\mathrm{D}t$. Due to the balance of momentum, (3.36), the term in paranthesis drops out. Also the term $\varepsilon_{ijk} u_j u_k$ cancels for symmetry reasons, and we are left with

$$\int_V \varepsilon_{ijk} T_{jk}\,\mathrm{d}V = 0 \;, \tag{3.46}$$

which, because of the arbitrary integration volume, is identical to the local statement

$$\varepsilon_{ijk} T_{jk} = 0 \quad . \tag{3.47}$$

Since ε_{ijk} is skew-symmetric in j and k, this implies that the stress tensor must be symmetric,

$$T_{ij} = T_{ji} \quad . \tag{3.48}$$

This important result shows that any material law for the tensor $\boldsymbol{T}$ must obey the relation $\boldsymbol{T} = \boldsymbol{T}^T$ in order to guarantee that the balance of angular momentum is satisfied.

3.5 Balance of mechanical energy

A balance equation for the mechanical energy, defined in (3.5), can be obtained by forming the dot product of the local momentum balance, (3.36), with $\boldsymbol{u}$.

This leads to

$$\begin{aligned}\rho\, u_i \frac{\mathrm{D}u_i}{\mathrm{D}t} = \rho\, \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{u_i u_i}{2}\right) &= \rho\, u_i k_i + u_i \frac{\partial T_{ji}}{\partial x_j} \\ &= \rho u_i k_i + \frac{\partial (u_i T_{ji})}{\partial x_j} - T_{ji}\frac{\partial u_i}{\partial x_j} \\ &= \rho u_i k_i + \frac{\partial (u_i T_{ji})}{\partial x_j} - T_{ij} S_{ij}\ ,\end{aligned} \tag{3.49}$$

where in the last step use was made of the symmetry of the stress tensor, $\boldsymbol{T}$. Integrating the local statement (3.49) over an arbitrary, but material, volume yields

$$\int_V \rho\, \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{u_i u_i}{2}\right) \mathrm{d}V = \int_V \left(\rho\, u_i k_i + \frac{\partial (u_i T_{ji})}{\partial x_j} - T_{ij} S_{ij}\right) \mathrm{d}V\ , \tag{3.50}$$

or, by use of (3.24) with $\boldsymbol{B} = (u_i u_i)/2$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\, \frac{u_i u_i}{2}\, \mathrm{d}V = \int_V \left(\rho u_i k_i - T_{ij} S_{ij} + \frac{\partial (u_i T_{ji})}{\partial x_j}\right) \mathrm{d}V\ , \tag{3.51}$$

where the left hand side corresponds to the change of the mechanical energy, $\mathrm{d}K/\mathrm{d}t$. It is instructive to re-formulate the last term on the right hand side in terms of a surface integral, leading to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\, \frac{u_i u_i}{2}\, \mathrm{d}V = \int_V (\rho u_i k_i - T_{ij} S_{ij})\, \mathrm{d}V + \int_A u_i t_i\, \mathrm{d}A\ , \tag{3.52}$$

where the relation (3.34) and Gauss' Theorem, (2.20), have been applied. This equation has a straighforward physical interpration. The change of mechanical energy of a material body, $\mathcal{B}$, is balanced by the rate of work done by the body forces, $\boldsymbol{k}$, the rate of work due to the surface stress, $\boldsymbol{t}$, and by the integral of the term

$$\mathcal{S} = -T_{ij} S_{ij}\ , \tag{3.53}$$

which is usually referred to as the *stress power*. In general, the stress power may be either a source or a sink of mechanical energy. Note that the balance of mechanical energy is not a first principle, because it can be derived without further assumptions from the balance of momentum.

3.6 Balance of total energy

The balance of total energy forms an additional first principle from which different statements can be derived. According to (3.12), the balance of energy for a material body, $\mathcal{B}$, can be written as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\left(\frac{u_i u_i}{2} + e\right) \mathrm{d}V &= \int_V \rho u_i k_i\, \mathrm{d}V + \int_V \rho r\, \mathrm{d}V \\ &+ \int_A u_i t_i\, \mathrm{d}A + \int_A h\, \mathrm{d}A\quad .\end{aligned} \tag{3.54}$$

The first term on the right hand side corresponds to the rate of work done by the body forces, and the second term to a supply of heat, where ρr denotes the supply of heat per unit volume. The third term is the rate of mechanical work done by the surface stress, and the last term denotes the rate of heat entering $\mathcal{B}$ through its surface.

Substracting the integral form of the balance of mechanical energy in (3.52) from (3.54) yields the integral form of *First Law of Thermodynamics* for a fluid:

$$\int_V \rho \frac{\mathrm{D}e}{\mathrm{D}t} \,\mathrm{d}V = \int_V (\rho r - T_{ij} S_{ij}) \,\mathrm{d}V + \int_A h \mathrm{d}A \;, \tag{3.55}$$

This fundamental equation states that the change in the internal energy of a material body corresponds to the sum of (a) the total amount of heat supplied through the terms involving r and h, and (b) the work performed against the internal stresses (or the "stress power", as noted above).

Similar to the previous sections, we now derive a local version of (3.55) by first attempting to convert the remaining surface integral on the right hand side into a volume integral. To this wend, we note that the amount, $h(\boldsymbol{x}, t; \boldsymbol{n})$, of heat entering per unit time through a surface element, $\boldsymbol{n}\mathrm{d}A$, does not only depend on the position of the surface element, but also on its orientation defined by the unit outward normal vector, $\boldsymbol{n}$. Similarly to the relation between the stress vector and the stress tensor, (3.34), it can be shown that $h(\boldsymbol{x}, t; \boldsymbol{n})$ is linear in $\boldsymbol{n}$,

$$h(\boldsymbol{x}, t; \boldsymbol{n}) = -q_i(\boldsymbol{x}, t) n_i \;, \tag{3.56}$$

where $\boldsymbol{q}(\boldsymbol{x}, t)$ is the vector of the heat flux. The physical interpration of (3.56) is that only the component of $\boldsymbol{q}$ perpendicular to the surface, i.e. parallel to the outward normal vector, $\boldsymbol{n}$, can exchange heat between the body and its environment. The minus sign in (3.56) appears because $\boldsymbol{n}$ points outward, whereas the heat flux is counted positive if directed towards the body.

With the help of (3.56), we can now use Gauss' Theorem to convert the surface integral in (3.55) into a volume integral. After rewriting the rate term on the left hand side with the help of (3.24), the internal energy budget in (3.55) can be re-expressed as

$$\int_V \left(\rho \frac{\mathrm{D}e}{\mathrm{D}t} - T_{ij} S_{ij} - \rho r + \frac{\partial q_i}{\partial x_i} \right) \mathrm{d}V = 0 \;, \tag{3.57}$$

which, for arbitrary intergration volumes and continuous fields, corresponds to the local statement

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = T_{ij} S_{ij} + \rho r - \frac{\partial q_i}{\partial x_i} \quad . \tag{3.58}$$

The physical interpration of this equation, which is the continuum mechanical equivalent of the *First Law of Thermodynamics*, is that the rate of change of internal energy, e, of a fluid particle equals the rate of mechanical work performed against the stresses, $T_{ij} S_{ij}$, plus a source term, ρr, and the divergence of the heat flux, q_i. The latter two thus correspond to the total amount of heat supplied to the fluid.

3.7 Rotating reference frame

Consider the motion of a point, P, in the Euclidian space illustrated in Figure 3.4. The position of this point can be described by the position vector, $\boldsymbol{r}(t)$, with respect to an inertial reference system, E, with base vectors $\boldsymbol{e}_i$. On the other hand, the motion of P can also be described by the position vector $\boldsymbol{x}(t)$ with respect to a moving reference system, E^*, with base vectors $\boldsymbol{e}_i^*(t)$. In general, the moving reference system will exhibit a translation, $\boldsymbol{l}(t)$, and a rotation, $\boldsymbol{\Omega}(t)$, with respect to the inertial system (see Figure 3.4).

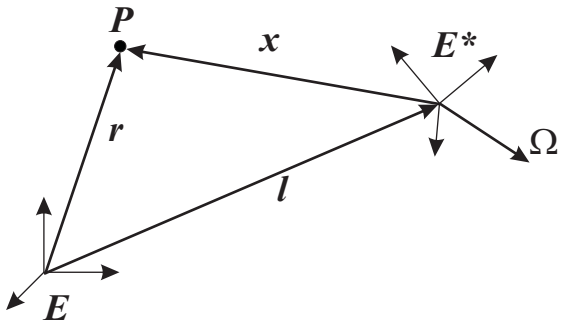

Figure 3.4: Position vectors of a point, P, in the inertial system, E, and the moving system, E^*.

It follows from the geometry shown in Figure 3.4 that the positions vectors obey the relation

$$\boldsymbol{r} = \boldsymbol{l} + \boldsymbol{x} = l_i \boldsymbol{e}_i + x_i^* \boldsymbol{e}_i^* \ , \tag{3.59}$$

where l_i are the components of the translation vector, $\boldsymbol{l}$, with respect to the inertial system, E, and x_i^* the components of the position vector, $\boldsymbol{x}$, with respect to the moving system, E^*.

Using the dot to indicate the time derivative, the velocity of P follows from the derivative of (3.59), and can be written as

$$\begin{aligned} \dot{r}_i \boldsymbol{e}_i &= \dot{l}_i \boldsymbol{e}_i + \dot{\overline{x_i^* \boldsymbol{e}_i^*}} \\ &= \dot{l}_i \boldsymbol{e}_i + \dot{x}_i^* \boldsymbol{e}_i^* + x_i^* \dot{\boldsymbol{e}}_i^* \ , \end{aligned} \tag{3.60}$$

where in the last line, due to the product rule or differentiation, the time derivative of a base vector, $\boldsymbol{e}_i^*$, appears. Since the base vector is, by definition, a unit vector, it can only change its direction, but not its length. In other words, it can only rotate around the axis of $\boldsymbol{\Omega}$ shown in Figure 3.4. If we consider, for a moment, the base vector as the position vector of a point, then the time derivative of the base vector can be regarded as the velocity of this point. This velocity, however, can only be the circumferential velocity due to the rotation of $\boldsymbol{e}_i^*$, and thus we conclude

$$\dot{\boldsymbol{e}}_i^* = \boldsymbol{\Omega} \times \boldsymbol{e}_i^* \quad . \tag{3.61}$$

Using this expression, the last line in (3.60) can be re-written as

$$\begin{aligned} \boldsymbol{u}^I = \dot{r}_i \boldsymbol{e}_i &= \dot{l}_i \boldsymbol{e}_i + \dot{x}_i^* \boldsymbol{e}_i^* + x_i^* \boldsymbol{\Omega} \times \boldsymbol{e}_i^* \\ &= \boldsymbol{u} + \boldsymbol{u}^l + \boldsymbol{\Omega} \times \boldsymbol{x} \ , \end{aligned} \tag{3.62}$$

which illustrates that the absolute velocity in the inertial system, $\boldsymbol{u}^I$, is the sum the velocity, $\boldsymbol{u} = \dot{x}_i^* \boldsymbol{e}_i^*$, measured in the moving system, the velocity of the system translation, $\boldsymbol{u}^l = \dot{l}_i \boldsymbol{e}_i$, and a contribution resulting from the rotation of the moving system.

In a similar way, an expression for the absolute acceleration of P can be derived. Starting with the time derivative of the first line of (3.62), and carrying out all derivatives according to the product rule, one finds

$$\begin{aligned} \boldsymbol{a}^I = \ddot{r}_i \boldsymbol{e}_i &= \ddot{l}_i \boldsymbol{e}_i + \ddot{x}_i^* \boldsymbol{e}_i^* + \dot{x}_i^* \dot{\boldsymbol{e}}_i^* \\ &+ \dot{x}_i^* \boldsymbol{\Omega} \times \boldsymbol{e}_i^* + x_i^* \dot{\boldsymbol{\Omega}} \times \boldsymbol{e}_i^* + x_i^* \boldsymbol{\Omega} \times \dot{\boldsymbol{e}}_i^* \\ &= \ddot{l}_i \boldsymbol{e}_i + \ddot{x}_i^* \boldsymbol{e}_i^* + \dot{x}_i^* \boldsymbol{\Omega} \times \boldsymbol{e}_i^* \\ &+ \dot{x}_i^* \boldsymbol{\Omega} \times \boldsymbol{e}_i^* + x_i^* \dot{\boldsymbol{\Omega}} \times \boldsymbol{e}_i^* + x_i^* \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{e}_i^*) \ , \end{aligned} \tag{3.63}$$

where again (3.61) has been used. Note that the time derivative of the reference frame rotation, $\dot{\boldsymbol{\Omega}}$, is identical in both, the inertial and the moving systems.

Collecting terms, (3.63) can be written as

$$\boldsymbol{a}^I = \boldsymbol{a} + \boldsymbol{a}^l + \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) \ , \tag{3.64}$$

which illustrates that the absolute acceleration, $\boldsymbol{a}^I$, measured in the inertial system, equals the sum of the acceleration, $\boldsymbol{a} = \ddot{x}_i^* \boldsymbol{e}_i^*$, measured in the moving system, the system acceleration, $\boldsymbol{a}^l = \ddot{l}_i \boldsymbol{e}_i$, and several contributions related to the rotation of the system, E^*.

This result is directly applicable to the balance of momentum, (3.36), if we identify the motion of the point, P, with the motion of a material point in the fluid. Since (3.36) is only valid in the inertial system, the velocity $\boldsymbol{u}$ in (3.36) clearly corresponds to the absolute velocity, $\boldsymbol{u}^I$, in (3.62), and the material time derivative, $\mathrm{D}\boldsymbol{u}/\mathrm{D}t$, in (3.36) corresponds to the absolute acceleration, $\boldsymbol{a}^I$, in (3.64). Then, (3.36) can be re-expressed as

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\rho \boldsymbol{a}^l - \rho \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - 2\rho\, \boldsymbol{\Omega} \times \boldsymbol{u} - \rho \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) + \rho \boldsymbol{k} + \nabla \cdot \boldsymbol{T} \ , \tag{3.65}$$

where $\mathrm{D}\boldsymbol{u}/\mathrm{D}t$ now denotes the acceleration of a material particle as measured in the moving system.

Several terms related to the motion of the reference system are seen to appear on the right hand side of (3.65), each of them having a particular physical interpretation. These terms are in the order of their appearance

1. the system acceleration, $-\ddot{\boldsymbol{l}}$, of the reference system
2. the acceleration due to the change of the angular velocity of the reference system, $-\dot{\boldsymbol{\Omega}} \times \boldsymbol{x}$,
3. the Coriolis acceleration, $-2\boldsymbol{\Omega} \times \boldsymbol{u}$, due to the velocity, $\boldsymbol{u}$, measured in the moving system,
4. and the centrifugal acceleration, $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x})$.

The components of the tensors in (3.65) can be expressed in term of the inertial system, E, or in terms of the moving system, E^*. We prefer the second possibility, since usually the motion of material points is described with respect to an observer in the moving system. Then, the component form of (3.65) reads

$$\begin{aligned} \rho \frac{\mathrm{D}u_i^*}{\mathrm{D}t} &= -\rho a_i^{l*} - \rho \epsilon_{ijk} \dot{\Omega}_j^* x_k^* - 2\rho \epsilon_{ijk} \Omega_j^* u_k^* - \rho \epsilon_{ijk} \epsilon_{lmk} \Omega_j^* \Omega_l^* x_m^* \\ &+ \rho k_i^* + \frac{\partial T_{ji}^*}{\partial x_j^*} \quad . \end{aligned} \tag{3.66}$$

The transformation rules between components in the E and E^* systems are identical to those discussed in the context of (1.14) and (1.16), with the important difference, however, that the transformation matrix,

$$a_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j^* \, , \tag{3.67}$$

now is a function of time.

It is interesting to note that, according to (3.61), the time derivative of a_{ij} is given by

$$\begin{aligned} \dot{a}_{ij} &= \boldsymbol{e}_i \cdot \dot{\boldsymbol{e}}_j^* \\ &= \boldsymbol{e}_i \cdot (\boldsymbol{\Omega} \times \boldsymbol{e}_j^*) \\ &= \boldsymbol{e}_i \cdot (\Omega_k^* \boldsymbol{e}_k^* \times \boldsymbol{e}_j^*) = \Omega_k^* \boldsymbol{e}_i \cdot (\epsilon_{kjl} \boldsymbol{e}_l^*) \\ &= -\epsilon_{ljk} \, \Omega_k^* a_{il} \quad . \end{aligned} \tag{3.68}$$

Multiplying both sides by $a_{im}/2$, and noting that, according to (1.10), $a_{il} a_{im} = \delta_{lm}$, (3.68) can be expressed as

$$A_{ij} = \frac{1}{2} \, \dot{a}_{mj} a_{mi} = -\frac{1}{2} \epsilon_{ijk} \Omega_k^* \, , \tag{3.69}$$

where it is clear from (2.51) that A_{ij} must be an anti-symmetric tensor. Using (2.50), this equation can be inverted to yield

$$\Omega_i^* = \epsilon_{ijk} A_{kj} = \frac{1}{2} \epsilon_{ijk} \dot{a}_{mj} a_{mk} \, , \tag{3.70}$$

which illustrates that $\boldsymbol{\Omega}$ can be computed from the transformation matrix, a_{ij}, and its derivative alone.

Chapter 4

Material laws

The balance equations discussed in the previous sections hold for material bodies of arbitrary material. The missing pieces of information to complete the thermo-mechanical description of a body are called *material laws*. The need for additional information becomes obvious by realizing that the first principles only yield 8 independent equations for the 17 unknowns ρ, u_i, e, T_{ij}, and q_i. This can bee seen by noting that there is 1 equation from the balance of mass, (3.15), 3 equations from the balance of momentum, (3.36), 3 independent equations from the balance of angular momentum, (3.47), and 1 equation from the balance of energy, (3.58). In addition, it is often required or useful to know the temperature, θ, which does not explicitly appear in the equations of motion.

The *theory of materials* attempts to supply the missing material laws by assuming, in the most general case, that the stress, $\boldsymbol{T}$, the heat flux, $\boldsymbol{q}$, and the internal energy, e, of a material point moving with the body may depend on the past and the present values of *all* field variables measured at *all* other material points in the body. The idea that the material behaviour of a body should only depend on the past and presence, and not on the future, is called the *Principle of Determinism*. The art in material theory is, of course, to find sufficiently general and precise relationships describing the numerous different classes of material, like elastic, plastic, viscous, etc. materials.

The above-mentioned completely general dependence of the material properties of material points on the properties of all other points considerably complicates a conclusive analysis of material laws. However, in view of the molecular structure of materials, it has been argued that it is plausible that information about the stress, the heat flux, and the internal energy at a material point should only come from a small region close to that point. Materials for which such a dependence on the *immediate neighbourhood* of a material point is in fact postulated are called *simple materials*. It is clear that in such materials the general dependence of $\boldsymbol{T}$, $\boldsymbol{q}$, and e, on all field variables reduces to a dependence on gradients, e.g. the velocity or temperature gradient, since gradients exactly contain the information about the immediate neighbourhood. We will restrict ourselves to such materials.

Even for deterministic and simple materials, the resulting expressions are still much to general to be of any practical use. It can be shown, however, that strong restrictions arise due to the fact that the material laws have to obey a number of important constraints.

1. The material laws have to be consistent with the balance laws found in the previous sections. They are, for example, not allowed to predict a non-symmetric stress tensor, $\boldsymbol{T}$, since then the balance of angular momentum would be violated. Neither are they allowed to predict that heat flows from cold to warm regions, since then the second law of thermodynamics would be violated.

2. They have to be valid in any coordinate system, implying that these laws have to be formulated in terms of tensorial quantities only.

3. They have to exhibit the *material symmetries* of the body. In crystals, for example, the molecular geometry determines how crystals deform for a given stress.

4. They must not depend on quantities related to the motion of a non-inertial reference system, i.e. they have to be of the same form in all reference systems. This property is called *material objectivity*.

The evaluation of the symmetry properties is the major ingredient for defining what a fluid distinguishes from other materials: in the sense of the theory of materials, fluids are *isotropic* materials, meaning that their material properties have no directional dependence at all. Fluid isotropy, among other things, implies that the deformation of the body itself does not have any influence on the stress, the heat flux, and the internal energy. This is in strong contrast to e.g. an elastic body, in which the stress is set exclusively by the deformation.

The last property, *material objectivity*, is perhaps the least obvious. However, it is of great importance for the derivation of material laws. To understand this property, consider, as an example, two observers measuring the force caused by the deformation of a spring without mass. One observer is assumed to live in a fixed, the other in a moving reference system. Material objectivity requires that, corresponding to our intuition, both observers measure the same relationship between force and deformation.

4.1 Thermodynamical equations of state

Before considering material laws for general bodies, it is instructive to first look at some classical results for *thermodynamical processes*, in which the thermodynamical variables are functions of the time only. In this section, we consider only homogeneous, isotropic fluids, in which the variables do not exhibit any dependence on the position. The *thermodynamical state* of such a fluid is described by a number of so-called *state variables*. Examples of such state variables are the density, ρ, the temperature, θ, the pressure, p, and many others. It is an

axiom of thermodynamics that for a one-component fluid, and we will restrict ourselves exclusively to such materials, any thermodynamical state variable is determined by at most two other state variables. Such a relation is called an *equation of state*. For a small class of simple materials, most notably gases and liquids of low molecular weight, these equations of state can be derived with the methods of statistical mechanics on the molecular level. In general, however, equations of state have to be found from experimental investigations.

An equation of state involving the thermodynamic variables p, ρ, and θ is called a *thermal equation of state*, for example

$$p = \hat{p}(\rho, \theta) \quad . \tag{4.1}$$

An explicit example is the *thermally ideal gas*, where (4.1) becomes

$$p = \rho R \theta \ , \tag{4.2}$$

with R being a constant, different for each gas.

Relations involving *caloric variables of state* like the internal energy, e, the *enthalpy*, $h = e + p/\rho$, or the entropy, s, are referred to as *caloric equations of state*. An example is

$$e = \hat{e}(\rho, \theta) = \tilde{e}(v, \theta) \ , \tag{4.3}$$

where, as indicated by the last term, sometimes the dependence on the density, ρ, is expressed in terms of the specific volume, $v = 1/\rho$. Note again, that this relation is only valid for fluids. For more general materials, the internal energy would obviously depend on additional parameters (like shear deformation contributing to elastic internal energy in elastic bodies).

Some interesting conclusions can be drawn from the total differential of the second of (4.3), which is

$$\mathrm{d}e = \left(\frac{\partial e}{\partial v}\right)_{\theta=\mathrm{const}} \mathrm{d}v \qquad + \quad \left(\frac{\partial e}{\partial \theta}\right)_{v=\mathrm{const}} \mathrm{d}\theta \quad . \tag{4.4}$$

If we assume that the flow is incompressible, $\mathrm{d}v = 0$, and introduce the function

$$c_v = \left(\frac{\partial e}{\partial \theta}\right)_{v=\mathrm{const}} \ , \tag{4.5}$$

we obtain from (4.4) the useful relation

$$\mathrm{d}e = c_v \mathrm{d}\theta \ , \tag{4.6}$$

where, in general, c_v is not constant. An interesting special case of (4.6) defines a *calorically ideal gas*, obeying the relation

$$e = c_v \theta \ , \tag{4.7}$$

where c_v now is a constant, depending on the type of the gas.

4.2 Stress tensor and heat flux

The first step in formulating material laws for $\boldsymbol{T}$, $\boldsymbol{q}$, and e is to identify the correct variables determining the behaviour of a certain material. These variables can be quite different for different materials. A first guess for an elastic body would be that stresses are caused by the *deformation* of the body, measured, for example, as the distance between two material points. For a viscous material, to the contrary, the deformation does not determine the stress. In this case, it is the time rate of deformation, measured, for example, as the velocity difference between two material points, that determines the stress. Similarly, we would identify the temperature difference between two points as a potential candidate for the heat flux. Keeping in mind that we consider only so-called simple materials, where only the neighbourhood of a material point can have an influence on the material behaviour, the above mentioned dependence on velocity and temperature differences reduces to a dependence on velocity and temperature gradients at the material point.

Continuing this procedure in a systematic way, and taking into account the fact that a fluid, by definition, is materially isotropic, the functional relationships for the stress, the heat flux, and the internal energy can be shown to be of a particular functional form. If we claim in addition that the fluid has no memory effects, in other words that the stress, the heat flux and internal energy are functions of the *present state* of the fluid only, it can be shown that this functional relationship must be of the form

$$\begin{aligned} \boldsymbol{T} &= \hat{\boldsymbol{T}}(\boldsymbol{S}, \theta, \nabla\theta, \rho) \\ \boldsymbol{q} &= \hat{\boldsymbol{q}}(\boldsymbol{S}, \theta, \nabla\theta, \rho) \\ e &= \hat{e}(\boldsymbol{S}, \theta, \nabla\theta, \rho) \ , \end{aligned} \tag{4.8}$$

where $\boldsymbol{S}$ is rate of deformation tensor, or the symmetric part of the velocity gradient. This is the most general form of a so-called *classical viscous heat conducting fluid*.

Complementary to (4.8) are the thermodynamical relations introduced in the Section 4.1, in particular the thermal and calorical equations of state. The important generalisation is that these relation, initially only defined for homogeneous fluids, are assumed to be valid for material volumes moving with the material particle.

From the theory of materials, the explicit functional form of (4.8) is known but, since it is somewhat complicated, we do not discuss it here. However, taking only the *linear* part of this general relationship, we obtain the *Cauchy-Poisson Law*,

$$T_{ij} = -p(\theta, \rho)\delta_{ij} + \mu^*(\theta, \rho)S_{kk}\delta_{ij} + 2\mu(\theta, \rho)S_{ij} \ , \tag{4.9}$$

where p is the *thermodynamic pressure*. The pressure comes into play because for the case of no deformation, $\boldsymbol{S} = \boldsymbol{0}$, the stress tensor in (4.9) becomes

$$T_{ij} = -p(\theta, \rho)\delta_{ij} \ , \tag{4.10}$$

and thus stresses can only be due to the pressure.

The functions μ^* and μ in (4.9) are, in general, functions of two variables of state, e.g. temperature and pressure as indicated. The function μ is usually referred to as the *shear viscosity*. Note that (4.9) predicts a symmetric stress tensor and thus is in agreement with the balance of angular momentum, (3.47).

The linear part of the general relation for the heat flux is called the *Fourier law*, postulating a linear relation between the heat flux and the temperature gradient,

$$q_i = -\lambda(\theta, \rho)\frac{\partial \theta}{\partial x_i} \ , \tag{4.11}$$

where the *heat conductivity*, λ, like the viscosities, is a function of two state variables.

Relations (4.9) and (4.11) satisfy all requirements for material laws. Fluids described by these material laws are called *Newtonian Fluids*. In spite of their simple description, these laws are in excellent agreement with the measured material properties of most gases and many fluids with low molecular weight, in particular for air, water, and most mineral oils. A consequence of this statement is that, after insertion of the material laws (4.9) and (4.11) in the balance equations, a set of equations can be obtained that describes virtually all flows in geophysics and engineering.

An interesting special case of (4.9) are incompressible fluids with

$$S_{ii} = \frac{\partial u_i}{\partial x_i} = 0 \ , \tag{4.12}$$

according to which the material law (4.9) simplifies to

$$T_{ij} = -p\delta_{ij} + 2\mu S_{ij} \quad . \tag{4.13}$$

This last equation shows that in an incompressible fluid at rest, the pressure is related to the trace of the stress tensor by

$$p = -\frac{1}{3}T_{ii} \quad . \tag{4.14}$$

4.3 The second law of thermodynamics

As we will see in this section, consistency with the *second law of thermodynamics* implies some important constraints for the material properties of a fluid that we have so far ignored. The second law is usually formulated in terms of a new thermodynamical state variable, called the entropy s per unit mass, that for the fluids investigated here is implicitly defined by the famous *Gibbs relation*:

$$T\mathrm{d}s = \mathrm{d}e + p\mathrm{d}v \ , \tag{4.15}$$

where T is the *absolute temperature*, and $v = \rho^{-1}$ the specific volume already introduced above. The Gibbs relation states that any change in internal energy

resulting from processes other than the work against the pressure is associated with a change in entropy. Using this definition, the second law of thermodynamics briefly outline in the context of (3.13) can be formulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho s \,\mathrm{d}V \geq \int_V \rho \frac{r}{T}\,\mathrm{d}V - \int_A \frac{q_i}{T} n_i \,\mathrm{d}A \;, \tag{4.16}$$

where, as above, V denotes the material volume of a body of fluid bounded by the material surface A. The expression in (4.16) states that the increase in entropy is always larger than the radiative entropy supply (first term on the right hand side) plus the surface entropy flux (second term). This statement forms a first principle of physics that is purely based on observational evidence, and cannot be proven.

Using (3.24) to move the time derivative on the left hand side in (4.16) under the integral, and (2.21) to convert the last integral into a surface integral, we find becomes

$$\int_V \left(\rho \frac{\mathrm{D}s}{\mathrm{D}t} - \rho \frac{r}{T} + \frac{\partial}{\partial x_i}\left(\frac{q_i}{T}\right)\right) \mathrm{d}V \geq 0 \quad . \tag{4.17}$$

Carying out the partial derivative in the last term, and using the usual argument about the arbitrariness of the integration volume, the local form of the entropy principle can be derived:

$$\rho T \frac{\mathrm{D}s}{\mathrm{D}t} - \rho r + \frac{\partial q_i}{\partial x_i} - \frac{1}{T}\frac{\partial T}{\partial x_i} q_i \geq 0 \quad . \tag{4.18}$$

This important relation, known as the *Clausius-Duhem inequality*, becomes more clearly by first rewriting the sum of the second and third terms with the help of the energy equation in (3.58):

$$-\rho r + \frac{\partial q_i}{\partial x_i} \;=\; -\rho \frac{\mathrm{D}e}{\mathrm{D}t} + S_{ij} T_{ij} \;, \tag{4.19}$$

For Newtonian fluids described by Cauchy-Poisson relation in (4.9), the *stress power* term appearing in (4.19) can be expressed as

$$\begin{aligned} S_{ij}T_{ij} &= -pS_{kk} + (\mu^* S_{ll} S_{mm} + 2\mu S_{ij} S_{ij}) \\ &= -p\frac{\partial u_i}{\partial x_i} + \Phi \\ &= \frac{p}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} + \Phi \\ &= -\rho p \frac{\mathrm{D}v}{\mathrm{D}t} + \Phi \;, \end{aligned} \tag{4.20}$$

where we have introduced the *dissipation function* Φ in the first step, used the balance of mass in the second, and the definition of the *specific volume* $v = \rho^{-1}$ in the third. Using this result, the heat supply and transport terms in (4.19) can be expressed as

$$-\rho r + \frac{\partial q_i}{\partial x_i} = -\rho \frac{\mathrm{D}e}{\mathrm{D}t} - \rho p \frac{\mathrm{D}v}{\mathrm{D}t} + \Phi \quad . \tag{4.21}$$

Inserting this relation into into (4.18), we obtain:

$$\rho\left(T\frac{\mathrm{D}s}{\mathrm{D}t}-\frac{\mathrm{D}e}{\mathrm{D}t}-p\frac{\mathrm{D}v}{\mathrm{D}t}\right)-\frac{1}{T}\frac{\partial T}{\partial x_i}q_i+\Phi\geq 0\;, \tag{4.22}$$

where, however, according to the Gibbs relation (4.15) the term in brackets vanishes. We therefore conclude that in general the entropy principle can only be satisfied if

$$-\frac{\partial T}{\partial x_i}q_i=\lambda\frac{\partial T}{\partial x_i}\frac{\partial T}{\partial x_i}\geq 0\;, \tag{4.23}$$

and

$$\Phi=\mu^*S_{kk}S_{ll}+2\mu S_{ij}S_{ij}\geq 0\;, \tag{4.24}$$

where we have used the Forier law in (4.11) and the definition of the dissipation function in (4.20). Note that the temperature θ and the absolute temperature T only differ by a constant factor, and can therefore be used interchangeably in all differential expressions. The expression in (4.23) indicates that if the heat flux is directed against the temperature gradient the entropy principle is violated. The necessary and sufficient conditions for this not to happen is $\lambda\geq 0$.

The inequality in (4.24) expresses the fact that work done against viscous forces is *irrersible*, which is in general only guaranteed for $\mu^*\geq 0$ and $\mu\geq 0$ (also this condition is necessary and sufficient). Therefore, if the equations of state connecting the material parameters λ, μ^*, and μ to other thermodynamical state variables are positive definite, the second law of the thermodynamics for a viscous, heat conducting fluid will never be violated, irrespective of the type of flow considered.

In this context, it should be recalled that according to (3.49), the stress power $S_{ij}T_{ij}$ quantifies the total conversion between mechanical and internal energy. The energy dissipation term Φ describes the *irreversible* part of this energy conversion, whereas the work performed against pressure forces, $-pS_{kk}$, is *reversible*.

Chapter 5

Newtonian and inviscid fluids

In this chapter, we specialize the universally valid balances of momentum, (3.36), and energy, (3.58), for Newtonian fluids. The balance of mass, (3.15), remains untouched by the material law, except for incompressible flows. The balance of angular momentum, (3.47), is automatically satisfied via the symmetry of the Cauchy-Poisson law, (4.9). Also the second law of thermodynamics cannot be violated, if (4.9) and (4.11) are used as material laws.

5.1 The Navier-Stokes equations

Inserting the Cauchy-Poisson law, (4.9), into the balance of momentum, (3.36), we obtain the *Navier-Stokes equations*,

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{\partial p}{\partial x_i} + \rho k_i + \frac{\partial}{\partial x_i}\left(\mu^* \frac{\partial u_k}{\partial x_k}\right) + \frac{\partial}{\partial x_j}\left[\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right] \quad . \tag{5.1}$$

With the help of the Cauchy-Poisson law, (4.9), and the Fourier law, (4.11), also the energy equation, (3.58), can be specialized for the case of Newtonian fluids,

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p\frac{\partial u_i}{\partial x_i} + (\mu^* S_{ii} S_{jj} + 2\mu S_{ij} S_{ij}) + \rho r + \frac{\partial}{\partial x_i}\left(\lambda \frac{\partial \theta}{\partial x_i}\right) \ , \tag{5.2}$$

where the special form of the stress power, $S_{ij}T_{ij}$, suggested in (4.20) has been used.

With the mass balance, (3.15), the Navier-Stokes equations, (5.1), and the energy equations for Newtonian fluids, (5.2), five equations are available for the seven unknowns ρ, θ, p, e, and u_i. The missing equations are two thermodynamic equations of state, for example of the form $p = \hat{p}(\rho, \theta)$ (required to compute the pressure gradient term) and $\theta = \hat{\theta}(\rho, e)$ (required to compute the

heat flux from the temperature gradient). The material parameters μ, μ^*, and λ depend on the thermodynamic state of the fluid, and can be expressed, for example, as functions of θ and ρ (or any other two state variables). The explicit functional forms have to be found from experiments.

This set of equations describes virtually all engneering and geophysical flows of water, gases, numereous mineral oils, and liquids of low molecular weight. Their solution, however, requires almost always numerical methods.

5.2 The Euler equations

If it is assumed that the fluid is inviscid, the momentum balance, (3.36), simplifies considerably. It might appear obvious at first glance that flows without friction are of little practical importance, but the contrary is the case. Many flows of technical and geophysical importance can be described with good accuracy by inviscid theory, among them many potential flows around wings and various wave phenomena like sound waves, surface gravity waves, internal waves. In many practical applications, viscous effects are important only in thin boundary layers. We will come back to this point in later chapters.

The special form of the momentum balance, (3.36), corresponding to inviscid fluids is called the *Euler equations*,

$$\rho \frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{\partial p}{\partial x_i} + \rho k_i \quad . \tag{5.3}$$

Thermodynamical arguments require that inviscid fluids also exhibit zero heat conductivity, $\lambda = 0$, leading to the energy equation of the form

$$\rho \frac{\mathrm{D}e}{\mathrm{D}t} = -p \frac{\partial u_i}{\partial x_i} + r \quad . \tag{5.4}$$

Equations (5.3) and (5.4) are accompagnied by the unchanged balance of mass, (3.15), leading to five equations for the six unkowns ρ, p, e, and u_i, such that one additional equation of state, for example of the form $p = \hat{p}(\rho, e)$, is required.

5.3 Incompressible flows

For incompressible flows, the balance of mass reduces to (3.17), which is also known as the *continuity equation*,

$$\frac{\partial u_i}{\partial x_i} = 0 \quad . \tag{5.5}$$

The stress tensor is given by (4.13), and the balance of momentum becomes

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + k_i + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \ , \tag{5.6}$$

where (5.5) has been used, and the *kinematic viscosity* or *diffusivity of momentum*, $\nu = \mu/\rho$, has been introduced. Like the viscosity, μ, also this quantity is, in principle, a function of two state variables, and thus a function of position. This spatial dependence, however, is in general of limited importance. Therefore, we assumed in the derivatation of (5.6) that μ is spatially constant.

For incompressible flows, the initial density distribution is often homogeneous, $\rho = \rho_0$. Since for incompressible flows, the density of a material particle does not change (see (3.16)), it follows that if the density distribution is homogeneous inititally, it is homogenous at all times. In this case, we can set $\rho = \rho_0$ in (5.6).

Also the balance of energy simplifies considerably, since for incompressible flows the internal energy, e, can be expressed in terms of the temperature, θ, according to (4.6). This leads to

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{2}{c_v}\nu S_{ij}S_{ij} + \frac{1}{c_v}r + \nu_\theta \frac{\partial^2\theta}{\partial x_i \partial x_i} \ , \tag{5.7}$$

where we introduced the *diffusivity of heat*, $\nu_\theta = \lambda/(\rho c_v)$. Like the diffusivity of momentum, ν, also the diffusivity of heat was assumed to be spatially constant in derving (5.7).

5.4 The Boussinesq assumption

In this section, we derive a special form of the Navier-Stokes equations that is a corner stone of geophysical modelling. Concentrating on geophysical flows, we first note that Newton's description of gravity, (3.11), can be simplified for bodies being close to the surface of the Earth. Then, the effects of gravity are described by the constant acceleration of gravity, $\boldsymbol{g}$, which is a vector pointing towards the center of the Earth. We identify the body force, $\boldsymbol{k}$, with this vector,

$$k_i = g_i = -g\delta_{i3} \ , \tag{5.8}$$

and ignore all other contributions to $\boldsymbol{k}$.

The key assumption in simplifying the Navier-Stokes equations in density stratified flows has been introduced by the French physicist and mathematician Valentin Joseph Boussinesq (1842-1929). He stated that the density, ρ, should be replaced by a constant *reference density*, ρ_0, everywhere, except for the gravity force term. The first consequence of this assumption is that the balance of mass, (3.15), adopts a form identical to that of incompressible fluids,

$$\frac{\partial u_i}{\partial x_i} = 0 \quad . \tag{5.9}$$

Using this relation, and Boussinesq's assumption, the balance of momentum becomes very similar to that for incompressible flows, (5.6), except for the body force term, and can be written as

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \frac{\rho}{\rho_0}g_i + \nu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad . \tag{5.10}$$

The energy balance, (5.2), is identical to that for an incompressible fluid, see (5.7).

The crucial advantage of the Boussinesq assumption is that the constancy of ρ in all terms but the buoyancy term removes sound waves. Sound waves are generally not of great interest in geophysical modelling, but, because they are fast and exhibit small spatial variations, there numerical treatment requires a tremendous amount of computational power.

5.5 Bernoulli equation

Particularly simple forms of the Euler equations, (5.3), can be obtained for two important special classes of flows. The first are irrotational flows, in which, as we saw before, the three components of the velocity are fully described by a scalar potential, $\boldsymbol{u} = \nabla\phi$. The second class are flows in which a "representative" streamline is known in advance. This is often the case for flows in tubes and ducts with variable cross section, in which the central streamline is determined by the geometry.

To derive equations describing these two classes of flows, we start by considering the advective term in the material derivative,

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} = \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \,, \tag{5.11}$$

appearing on the left hand side of (5.3). The advective term can be re-written as

$$\begin{aligned} u_j \frac{\partial u_i}{\partial x_j} &= u_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) + u_j \frac{\partial u_j}{\partial x_i} \\ &= 2u_j W_{ij} + \frac{\partial}{\partial x_i} \left(\frac{u_k u_k}{2} \right) \,, \end{aligned} \tag{5.12}$$

where in the last step we used the definition (2.36) of the skew-symmetric vorticity tensor, $\boldsymbol{W}$. Since this tensor, according to (2.51), is uniquely determined by the rotation, $\boldsymbol{\omega}$, of the velocity field, (5.12) can also be written as

$$\begin{aligned} u_j \frac{\partial u_i}{\partial x_j} &= -\epsilon_{ijk} u_j \omega_k + \frac{\partial}{\partial x_i} \left(\frac{u_k u_k}{2} \right) \\ &= -\epsilon_{ijk} u_j \omega_k + \frac{1}{2} \frac{\partial u^2}{\partial x_i} \end{aligned} \tag{5.13}$$

where in the last step we introduced the abbreviation $u^2 = u_k u_k$.

With the help of (5.13), the Euler equations, (5.3), become

$$\frac{\partial u_i}{\partial t} - \epsilon_{ijk} u_j \omega_k + \frac{1}{2} \frac{\partial u^2}{\partial x_i} + \frac{1}{\rho} \frac{\partial p}{\partial x_i} - k_i = 0 \,, \tag{5.14}$$

which are the starting point of all results derived in this section.

5.5.1 Bernoulli equation for irrotational flows

Now we consider, as the first special case, irrotational flows in which the velocity field is fully described by a velocity potential according to $u_i = \partial\phi/\partial x_i$. Additionally, let us assume that also the external forces can be described by a potential according to $k_i = -\partial\psi/\partial x_i$. As example, we consider the potential of gravity, $\psi = -g_k x_k$, such that $k_i = g_i$ coincides with the acceleration of gravity.

Since in irrotational flows, $\boldsymbol{\omega} = 0$ by definition, the second term in (5.14) vanishes. Then, using the potentials of the velocity, ϕ, and body force, $\psi = -g_k x_k$, the relation (5.14) reads

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\frac{\partial\phi}{\partial x_k}\frac{\partial\phi}{\partial x_k} - g_k x_k\right) + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 \quad . \tag{5.15}$$

Let us now have a closer look at the pressure gradient term in (5.14). If this term could be written as

$$\frac{1}{\rho}\frac{\partial p}{\partial x_i} = \frac{\partial P}{\partial x_i} \, , \tag{5.16}$$

where P is a yet unkown function, then the pressure term in (5.15) could be moved into the brackets, and the whole equation could be integrated in space. Unfortunately, a generally valid expression for P does not exist. However, if we only consider fluids in which the density is a function of the pressure, $\rho = \hat{\rho}(p)$, then, according to (5.16), the total differential,

$$\mathrm{d}P = \frac{1}{\rho}\mathrm{d}p \, , \tag{5.17}$$

only depends on the pressure as well. Such fluids are called *barotropic*, and include constant-density fluids with $\rho = \rho_0$ as a special case. For a baroropic fluid, it is straightfoward to obtain the *pressure function* P by integrating (5.17), which yields

$$P = \int_0^p \frac{1}{\rho(p')}\mathrm{d}p' \quad . \tag{5.18}$$

The point in this definition is that the gradient of P can be written as

$$\frac{\partial P}{\partial x_i} = \frac{\mathrm{d}P}{\mathrm{d}p}\frac{\partial p}{\partial x_i} = \frac{1}{\rho}\frac{\partial p}{\partial x_i} \, , \tag{5.19}$$

and thus corresponds exactly to the pressure gradient term in (5.15). Using this result, we can re-express (5.15) as

$$\frac{\partial}{\partial x_i}\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_i} + P - g_i x_i\right) = 0 \, , \tag{5.20}$$

in which now the whole left hand side is expressed in gradient form. This implies that (5.20) can be integrated along an *arbitrary* curve, $\mathcal{C}$, yielding

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_i} + P - g_i x_i = C(t) \quad . \tag{5.21}$$

where the right hand side is in general a function of time.

We now re-define the velocity potential as

$$\tilde{\phi} = \phi - \int_0^t C(t')\mathrm{d}t' \ , \tag{5.22}$$

which, since we still have $\boldsymbol{u} = \nabla\tilde{\phi}$, does not imply any loss of generality. The advantage of the transformation is that we can now express (5.21) as

$$\frac{\partial\tilde{\phi}}{\partial t} + \frac{1}{2}\frac{\partial\tilde{\phi}}{\partial x_i}\frac{\partial\tilde{\phi}}{\partial x_i} + P - g_i x_i = 0 \ , \tag{5.23}$$

where the dependence on $C(t)$ has dissappeared. This equation is the famous *Bernoulli Equation* for potential flows. The advantage gained with (5.23) compared to the original Euler equations, (5.3), cannot be overemphasized: the Euler equations are a non-linearly coupled system of partial differential equations, whereas the Bernoulli equation is just one differential equation for the scalar field $\tilde{\phi}$. This equation plays a central role in accoustics and, as we will see below, also in the description of surface waves.

5.5.2 Bernoulli equation for flows with known streamline

As mentioned above, another version of the Bernoulli equation can be derived in flows for which the geometry of streamlines is already known. To see this, we start again from (5.14), assuming as before that the flow is barotropic and that the potential of the body forces, $\boldsymbol{k}$, corresponds to the potential of gravity. Then, (5.14) becomes

$$\frac{\partial u_i}{\partial t} - \epsilon_{ijk}u_j\omega_k + \frac{\partial}{\partial x_i}\left(\frac{u^2}{2} + P - g_i x_i\right) = 0 \quad . \tag{5.24}$$

Now, we consider the projection of this equation in the direction of an element, $\mathrm{d}\boldsymbol{x}$, of the streamline. This projection follows simply from the scalar product of (5.24) with $\mathrm{d}\boldsymbol{x}$,

$$\frac{\partial u_i}{\partial t}\,\mathrm{d}x_i - \epsilon_{ijk}u_j\omega_k\,\mathrm{d}x_i + \frac{\partial}{\partial x_i}\left(\frac{u^2}{2} + P - g_i x_i\right)\mathrm{d}x_i = 0 \quad . \tag{5.25}$$

Since $\mathrm{d}x_i$ is a line element of a streamline, it obeys the relation $\mathrm{d}x_i/\mathrm{d}s = u_i/u$ according to (2.8). With the help of this relation, the first term in (5.14) becomes

$$\frac{\partial u_i}{\partial t}\,\mathrm{d}x_i = \frac{1}{u}\frac{\partial u_i}{\partial t}u_i\mathrm{d}s = \frac{1}{u}\frac{1}{2}\frac{\partial u^2}{\partial t}\mathrm{d}s = \frac{\partial u}{\partial t}\mathrm{d}s \ , \tag{5.26}$$

where again we have used the abbreviation $u^2 = u_i u_i$.

For the rotation term in (5.25), we obtain

$$\epsilon_{ijk}u_j\omega_k\,\mathrm{d}x_i = \frac{1}{u}\epsilon_{ijk}u_i u_j\omega_k\mathrm{d}s = 0 \ , \tag{5.27}$$

where the last step follows from the fact the ϵ_{ijk} is anti-symmetric in i and j, whereas $u_i u_j$ is symmteric.

By use of (5.26) and (5.27), relation (5.25) becomes

$$\frac{\partial u}{\partial t}\,\mathrm{d}s + \frac{\partial}{\partial x_i}\left(\frac{u^2}{2} + P - g_i x_i\right)\mathrm{d}x_i = 0\ , \tag{5.28}$$

which can be integrated along the streamline, $\mathcal{C}$, to yield

$$\int_{\mathcal{C}} \frac{\partial u}{\partial t}\,\mathrm{d}s + \frac{u^2}{2} + P - g_i x_i = \bar{C}\ , \tag{5.29}$$

where $\bar{C}$ is a constant that is, in general, different for different streamlines. In stationary flows, the integral in (5.29) vanishes, and we obtain

$$\frac{u^2}{2} + P - g_i x_i = \bar{C} \quad . \tag{5.30}$$

Thus, (5.30) constitutes a relation between values of variables located at any two points on the streamline, independent of their values between these points.

Another result of great practical importance can be obtained by observing that for stationary and irrotational flows the first two terms in (5.25) are zero by definition, and we can integrate this equation following an *arbitrary* integration path. The formal result of this integration coincides with (5.30), with the important difference that the constant of integration now is idendentical throughout the flow. In this case, (5.30) constitutes a relation between the values of u, P, and the potential $g_i x_i$ between two *arbitrary* points in the flow.

5.6 Boundary conditions

The Navier-Stokes equations, (5.1) and (5.2), are a coupled set of partial differential equations. To solve these equations, a set of appropriate boundary conditions is required. Two different groups of boundary conditions need to be statisfied.

The first group of these boundary conditions are called physical boundary conditions. They can, for example, be formulated to prescribe the *value* of the dependent variable explicitly at the boundary, for example the velocity or the temperature. Alternatively, one can also attempt to prescribe the *gradient* of the variable normal to the boundary, which often can be evaluated from the known flux of the variable across the boundary, for example the momentum flux (stress) or the heat flux.

The second group of boundary conditions results from the purely kinematic constraint that the bounding surface of a material body has to be a material surface. This type of boundary conditions is called kinematic boundary conditions, and is completely independent from the physical boundary conditions mentioned above.

5.6.1 Physical boundary conditions

At a solid boundary, the velocity of a Newtonian fluid is assumed to correspond to the velocity, $\boldsymbol{u}_B$, of the boundary,

$$\boldsymbol{u} = \boldsymbol{u}_B \quad \text{at the boundary.} \tag{5.31}$$

In particular, for a solid boundary at rest, we have $\boldsymbol{u} = \boldsymbol{0}$. Note that for inviscid fluids, the component tangential to the boundary does in general not correspond to the tangential velocity of the boundary. Only the normal components have to corresponds,

$$\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{u}_B \cdot \boldsymbol{n} \quad \text{at the boundary,} \tag{5.32}$$

which is a manifestation of the fact the Euler equations are only first order in space, and cannot satisfy two independent boundary conditions at the same boundary point.

Similarly to (5.31), the internal energy of the fluid, has to correspond to the internal energy, e_B, at the boundary,

$$e = e_B \quad \text{at the boundary} \quad . \tag{5.33}$$

The internal energy can for example be evaluated by an equation of state from the temperature and the density at the boundary, but equally well from any other pair of state variables.

For an incompressible or a Boussinesq fluid, this boundary condition is replaced by a boundary condition for the temperature equation, (5.7), by simply prescribing the temperature,

$$\theta = \theta_B \quad \text{at the boundary} \quad . \tag{5.34}$$

The boundary conditions (5.31), (5.33), and (5.34) directly prescribe the value of the dependent variables at the boundary and thus correspond to *Dirichlet* boundary conditions.

Often, however, it is not the value of the indenpendent variable at the boundary that is known, but the flux of the variable. Boundary conditions that can be derived from this information are often called *Neumann*-type boundary conditions. Their use is most obvious, if the amount of heat, h, entering the fluid across the boundary is considered, as it is the case in many technical applications. If $\boldsymbol{n}$ is the outward unit normal vector at the boundary, the amount of heat entering the body is

$$h = h_B = q_i n_i = -\lambda \frac{\partial \theta}{\partial x_i} n_i = -\lambda \frac{\partial \theta}{\partial n_i} \quad \text{at the boundary ,} \tag{5.35}$$

where the Fourier law, (4.11), for the heat flux has been used. The last relation consitutes a boundary condition for the derivative of the independent variable. Likewise, the flux of momentum (or stress) can be specified at the boundary

$$\boldsymbol{t} = \boldsymbol{t}_B = \boldsymbol{T}\boldsymbol{n} \quad \text{at the boundary} \quad . \tag{5.36}$$

For inviscid flows, according to (4.9), this condition becomes

$$\boldsymbol{p} = \boldsymbol{p}_B \quad \text{at the boundary} \quad . \tag{5.37}$$

Note that in the case of moving or open boundaries, also advective fluxes of heat and momentum have to be taken into consideration.

5.6.2 Kinematic boundary condition

The so-called kinematic boundary condition results from the simple kinematic constraint that a material point located at a boundary at $t = t_0$ stays at the boundary for all times. In other words, a material body is always bounded by a material surface. This statement is sometimes called the *Theorem of Lagrange*. Some interesting consequences follow from it.

An arbitrary surface in E_3 can be specified by an implicit relation of the form

$$F = \hat{F}(\boldsymbol{x}, t) = 0 \ , \tag{5.38}$$

which has to be satisfied for all times for points located on the surface. If the surface is a material surface, it consists, by definition, of the same material points for all times. Every material point has to satisfy (5.38) for all times, i.e. the change of F, as seen from the material points, is zero. Since the change of a variable seen by moving material point is described by the material derivative, this statement is equivalent to

$$\frac{\mathrm{D}\hat{F}}{\mathrm{D}t} = \frac{\partial \hat{F}}{\partial t} + \boldsymbol{u} \cdot \nabla \hat{F} = 0 \quad . \tag{5.39}$$

This equation has some interesting implications. First, we recall that the gradient of $\hat{F}$ is parallel to the unit normal vector on the material surface,

$$\nabla \hat{F} = \boldsymbol{n} \left| \nabla \hat{F} \right| \ , \tag{5.40}$$

where $\boldsymbol{n}$ is not necessarily pointing outward. This relation can be used to rewrite (5.39) in terms of the normal velocity, $\boldsymbol{u} \cdot \boldsymbol{n}$, as

$$\frac{\partial \hat{F}}{\partial t} + \boldsymbol{u} \cdot \nabla \hat{F} = \frac{\partial \hat{F}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{n} \left| \nabla \hat{F} \right| = \frac{\partial \hat{F}}{\partial t} + \boldsymbol{u}_B \cdot \boldsymbol{n} \left| \nabla \hat{F} \right| = 0 \ , \tag{5.41}$$

where we used the fact that, at the boundary, the normal velocity of the body has to be equal to the normal velocity of the boundary, $\boldsymbol{u}_B \cdot \boldsymbol{n}$, if the boundary is impermeable, see (5.32). The last expression in (5.41) leads to

$$\frac{\partial \hat{F}}{\partial t} + \boldsymbol{u}_B \cdot \nabla \hat{F} = 0 \ , \tag{5.42}$$

which is a special form of (5.39), valid only if the bounding surface is an impermeable wall.

Note that relation (5.41) may be re-written as

$$\boldsymbol{u}_B \cdot \boldsymbol{n} = -\left|\nabla \hat{F}\right|^{-1} \frac{\partial \hat{F}}{\partial t} \ , \tag{5.43}$$

yielding a convenient expression for the normal velocity of a moving surface.

In contrast to (5.42), the kinematic boundary condition, (5.39), is also valid for free surfaces. The additional complication then, however, is that both speed and position of the free surface are part of the solution, and not prescribed.

In geophysical applications it is often possible to describe bounding surfaces as *explicit* functions, in contrast to (5.38), corresponding to an implicit description. An example is the free surface of open water bodies like rivers, lakes, and the ocean. Taking z as the coordinate pointing upwards, with $z = 0$ being an arbitrary reference level (for example the mean water level), the position of the free surface at every horizontal position can be expresed by the function

$$z = \eta(x, y, t) \quad . \tag{5.44}$$

This description fails if the surface is multivalued with respect to the vertical coordinate, for example in the case of breaking waves. Evidently, the explicit description (5.44) corresponds to the implicit description

$$F = F_s(x, y, z, t) = \eta(x, y, t) - z = 0 \quad . \tag{5.45}$$

Analogously, we can describe the bottom topography of the ocean by the explicit function

$$z = -H(x, y) \ , \tag{5.46}$$

where $H(x, y)$ corresponds to the stationary bottom topography. Again, this explicit description fails for multivalued topographies with overhangs or caves. The implicit equivalent to (5.46) is

$$F = F_b(x, y, z,) = H(x, y) + z = 0 \quad . \tag{5.47}$$

The kinematic boundary condition, (5.39), at the free surface leads to

$$\frac{\mathrm{D}F_s}{\mathrm{D}t} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} - w = 0 \quad \text{at } z = \eta(x, y, t) \quad . \tag{5.48}$$

Likewise, at the bottom we have

$$\frac{\mathrm{D}F_b}{\mathrm{D}t} = u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + w = 0 \quad \text{at } z = -H(x, y) \quad . \tag{5.49}$$

We will use these relations frequently in later chapters.

Chapter 6

Gravity waves

Waves are an ubiquitous feature in technical and geophysical fluid motion. The essential property of a wave-like motion is that energy and phase of some disturbance are transported from one point to another, with the important constraint that the speed of the medium guiding the wave is generally small compared to the wave speed. Waves occur typically as the response of a system to an inital disturbance that is "fast" in some sense. In fluids, numerous different types of waves are known. One type of wave motions, found in compressible fluids only, are *compression waves* in which disturbances are transmitted in form of pressure or density signals in the medium. The small amplitude version of these waves is well known as a *sound wave*.

Another type of wave motion occurs at the free surface of a fluid in a gravity field. These *surface gravity waves* are visible as moving displacements of the free surface. Individual crests and troughs, indicating the *phase* of the wave, travel with the *phase speed*, where the local distance between two crests (or troughs) is the *wave length*. Even though the phase speed can be quite large in some types of surface gravity waves, the time-averaged motion of the fluid due to the presence of waves is always very small.

6.1 Plane harmonic waves

Waves can have many different geometries. For example, sound waves generated from a point source in a three-dimensional homogeneous and isotropic medium exhibit a *spherical* geometry. Surface gravity waves caused by a point source, e.g. a stone thrown into the water, have a *cylindrical* geometry. The most simple geometry of waves, however, corresponds to *plane waves*, in which surfaces of constant phase are planes. Examples are sound waves near a large plane source of sound (e.g. a large loud speaker) or surface gravity waves generated by a plane wave maker. Also surface gravity waves generated by a homogeneous wind field are prescribed by a plane wave. Plane waves are often a useful local approximation for more complex wave geometries in a certain limit: far from

the source, a cylindrical or spherical wave can often be locally approximated by a plane wave.

Geometrically, a point, $\boldsymbol{x}$, (not material a point) located on a moving plane in E_3 has to satisfy the implicit relation

$$\boldsymbol{x} \cdot \boldsymbol{n} = a(t) \,, \tag{6.1}$$

where $\boldsymbol{n}$ is the unit vector normal to the plane, and a an arbitrary function of time. The time derivative of (6.1) yields

$$\boldsymbol{c} \cdot \boldsymbol{n} = c \,, \tag{6.2}$$

where $c = |\boldsymbol{c}|$ denotes the speed of the plane in the direction of $\boldsymbol{n}$. If we assume that c is constant in time, it follows from integration that

$$a(t) = ct + \phi \,, \tag{6.3}$$

where ϕ is a constant of integration, called the *phase* of the wave. Assuming that c does not depend on $\boldsymbol{x}$ and $\boldsymbol{n}$ means that waves move in a *homogeneous* and *isotropic* medium, respectively.

Inserting (6.3) in (6.1) yields

$$\boldsymbol{x} \cdot \boldsymbol{n} - ct = \phi \,, \tag{6.4}$$

indicating that for points moving with the phase plane, the phase, ϕ, does not change. A wave crest of a surface wave, for example, moves with the phase speed inside a phase plane perpendicular to the mean water surface, such that it always corresponds to the same phase.

So far we haven't said anything about the geometry of the plane waves in their direction of propagation. Whether, for example, the waves are sinusoidal or step-like shock waves, is determined by a scalar function of the phase,

$$F = \hat{F}(\phi) = \hat{F}(\boldsymbol{x} \cdot \boldsymbol{n} - ct) \quad . \tag{6.5}$$

In case of shock waves, for example, the function F would correspond to the Heaviside step function. Then, (6.5) would describe a step travelling in the direction of $\boldsymbol{n}$ with constant speed c. Often observed are also periodic waves, for which $F = F_P$ corresponds to a periodic function with period P, leading to

$$F_P = \hat{F}_P \left(\frac{1}{k} \left(\boldsymbol{x} \cdot k\boldsymbol{n} - kct \right) \right) = \tilde{F}_P \left(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t \right) \quad . \tag{6.6}$$

Here, we have introduced the *circular frequency*,

$$\omega = kc = \frac{P}{T} \,, \tag{6.7}$$

where T is the *wave period*. Note that this is different from the classical definition of the frequency $f = 1/T$. Also introduced was the *wave vector*, $\boldsymbol{k}$, pointing

into the direction of propagation and defining the *wave number*, $k = |\boldsymbol{k}|$. Similarly to (6.7), the wave number is defined by

$$k = \frac{P}{\lambda} \, , \tag{6.8}$$

and thus is a local measure of the density of wave crests per unit length.

A very important special case of (6.6) are harmonic plane waves described by

$$F_P = e^{i(\boldsymbol{x}\cdot\boldsymbol{k}-\omega t)} = \cos\left(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t\right) + i\sin\left(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t\right) \quad . \tag{6.9}$$

for which, because the period of the harmonic function F_P is $P = 2\pi$, the wave circular frequency and number obey

$$\omega = \frac{2\pi}{T} \, , \quad k = \frac{2\pi}{\lambda} \quad . \tag{6.10}$$

6.2 Formulation of the problem

In the following we consider only plane harmonic surface waves. The coordinate system is chosen such that the z-axis points upward with $z = 0$ corresponding to the undisturbed water level, and the x-axis is aligned with the direction of propgation, i.e. with the wave vector, $\boldsymbol{k}$.

Considering only surface gravity waves in unstratified fluids, a number of simplifications of the equations of motion are possible. The most important simplifications arises from the observations that surface waves are described very accurately by assuming that the flow is irrotational and that viscous effects can be ignored. This implies that surface wave motion is a potential flow that can be described by the Bernoulli equation for irrotational flows. Another very useful observation is that compressibility effects in surface wave motion are irrelevant. In incompressible flows the balance of mass reduces to the continuity equation, (5.5), which, for the two-dimensional geometry considered, becomes

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \quad . \tag{6.11}$$

Introducing the velocity potential defined in (2.55), this equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \, , \tag{6.12}$$

which is the *Laplace equation*, the perhaps most well-known partial differential equation.

The Bernoulli equation for irrotational flows, (5.23), becomes

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left(\frac{\partial \phi}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial z}\right) + \frac{p}{\rho_0} + gz = 0 \, , \tag{6.13}$$

where the pressure function, P, has already been evaluated by assuming that the flow is incompressible and, additionally, the density, ρ_0, is constant.

The boundary condition for the stress at the surface in an inviscid fluid is given by (5.37), which becomes

$$p = 0 \quad \text{at } z = \eta \,, \tag{6.14}$$

if we assume that the atmospheric pressure is dynamically negligible everywhere at the free surface.

Further, assuming that both free surface and bottom are described by single-valued, explicit functions, the kinematic boundary conditions are can be specified by using (5.48) and (5.49). For a potential flow with the coordinate system described above, these boundary conditions become

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} - \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = \eta(x,t) \,, \tag{6.15}$$

and

$$\frac{\partial \phi}{\partial x}\frac{\partial H}{\partial x} + \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -H(x) \quad . \tag{6.16}$$

This set of equations and boundary conditions yields a prescise description of almost all aspects of inviscid wave motion. Unfortunately, the solution of the equations is not easy for the following two reasons. First, the Bernoulli equation, (6.13), and the kinematic boundary conditions, (6.15) and (6.16), contain non-linear terms. Non-linearities strongly reduce the chance to find closed analytical solutions to a set of partial differential equations. Second, the surface boundary conditions, (6.14) and (6.15) have to be imposed at the free surface, $z = \eta(x,t)$, whose position is part of the solution and thus unkown. It is not obvious how these boundary conditions should be applied in the solution procedure. For these reasons, usually a number of additional assumptions is introduced, mainly in order to obtain a *linear* system of equations with surface boundary conditions imposed at a *fixed* position. The resulting equations still yield a fairly detailed and accurate description of many surface wave phenomena.

For the linearisation of the differential equations, it is essentially assumed that fluctuations of physical quantities like velocity and surface excursions caused by the presence of waves are small. All non-linearities in the equations described are quadratic, i.e. arise from the product of two quantities. The product of two small quantities of the order ϵ is of the order ϵ^2, which is very small and can be neglected.

It can be shown that the essential assumptions for the linearisation are

$$a \ll H \,, \quad a \ll \lambda \,, \tag{6.17}$$

where a denotes the amplitude of the free surface excursions. This approximation implies that the waves are not steep ($a \ll \lambda$) and that the undisturbed water depth is not significantly deviate from the instantaneous water depth ($a + H \approx H$).

Omitting the non-linear term in parentheses in (6.13) leads to the linear form of the Bernoulli equation,

$$\frac{\partial \phi}{\partial t} + \frac{p}{\rho_0} + gz = 0 \quad . \tag{6.18}$$

Applying this equation at any point at the surface, and inserting the boundary condition (6.14), the pressure can be removed from the problem. The resulting new boundary condition reads

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = \eta \quad . \tag{6.19}$$

After ignoring the non-linear terms in the kinematic boundary conditions, these can be written as

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = \eta(x,t) \; , \tag{6.20}$$

and

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -H \; , \tag{6.21}$$

where from here on we assume for simplicity that the bottom is flat.

This set of equations is now linear and thus much easier to handle. However, the new physical boundary condition, (6.19), as well as the kinematic boundary condition, (6.20), still have to be imposed at the (unkown) free surface. In particular, the term $\partial\phi/\partial t$ in (6.19) and the term $\partial\phi/\partial z$ in (6.20) have to evaluated at the unkown position $z = \eta(x,t)$.

Another type of linearisation helps to overcome this problem. To see this, we first note that, for a fixed time, $\partial\phi/\partial t$ and $\partial\phi/\partial z$ are smooth functions of z, and thus can be expanded in a Taylor-series. For the latter term, this yields

$$\begin{aligned} \frac{\partial \phi}{\partial z}\Big|_{z=\eta} &= \frac{\partial \phi}{\partial z}\Big|_{z=0} + \eta \, \frac{\partial^2 \phi}{\partial z^2}\Big|_{z=0} + \dots \\ &\approx \frac{\partial \phi}{\partial z}\Big|_{z=0} \; , \end{aligned} \tag{6.22}$$

where in the last step the lowest order of approximation has been used in neglecting the linear term. Similarly, it can be shown that

$$\frac{\partial \phi}{\partial t}\Big|_{z=\eta} \approx \frac{\partial \phi}{\partial t}\Big|_{z=0} \quad . \tag{6.23}$$

Thus, approximated versions of the surface boundary conditions (6.19) and (6.20) can be written as

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = 0 \; , \tag{6.24}$$

and

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = 0 \; , \tag{6.25}$$

which are now evaluated at the fixed position $z = 0$, instead of the free surface.

6.3 Solution for harmonic waves

Summarizing the results derived above, we have to solve the two dimensional continuity equation, (6.12), for potential flows,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \,, \tag{6.26}$$

which corresponds to the linear Laplace equation for the potential ϕ. The linearised boundary conditions for this partial differential equation are the dynamical boundary condition

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = 0 \,, \tag{6.27}$$

and the two kinemtic boundary conditions to be satisfied at the surface and the bottom,

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \quad \text{at } z = 0 \,, \tag{6.28}$$

and

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = -H \,, \tag{6.29}$$

respectively.

We look for harmonic plane waves solutions for this set of equations. For the surface displacement, plane waves travelling in the x-direction are described by

$$\eta(x,t) = a\cos\left(kx - \omega t\right) \quad . \tag{6.30}$$

For the velocity potential, we use a similar Ansatz of the form

$$\phi(x,z,t) = f(z)\sin(kx - \omega t) \,, \tag{6.31}$$

where the vertical variation of ϕ is assumed to be fully described by the function $f(z)$. The appearence of the sine is obvious from the cosine in (6.30) and the derivative in (6.27).

Insertion of (6.31) into (6.26) yields

$$\frac{\partial^2 f}{\partial z^2} - k^2 f = 0 \,, \tag{6.32}$$

which is a linear, homogenous differential equation with the well-known solution

$$f(z) = Ae^{kz} + Be^{-kz} \tag{6.33}$$

with unkown constants A and B.

Using this result, (6.31) can be written as

$$\phi(x,z,t) = \left(Ae^{kz} + Be^{-kz}\right)\sin(kx - \omega t) \,, \tag{6.34}$$

containing the four unknowns A, B, k and ω. With the three boundary conditions which we have not yet used, we expect to find solutions for A, B and a

relation between k and ω. Insertion of the bottom boundary condition, (6.29), into (6.34) leads to

$$B = Ae^{-2kH}. \tag{6.35}$$

Insertion of (6.30) and (6.34) into (6.28) leads to

$$k(A - B) = a\omega. \tag{6.36}$$

Combining (6.35) and (6.36) gives

$$A = \frac{a\omega}{k\,(1 - e^{-2kH})}\,, \quad B = \frac{a\omega e^{-2kH}}{k\,(1 - e^{-2kH})} \quad . \tag{6.37}$$

Introducing the hyperbolic functions $\cosh(x) = \frac{1}{2}(\exp(x) + \exp(-x))$ and $\sinh(x) = \frac{1}{2}(\exp(x) - \exp(-x))$, we finally obtain the potential

$$\phi = \frac{a\omega}{k}\frac{\cosh(k(z + H))}{\sinh(kH)}\sin(kx - \omega t), \tag{6.38}$$

from which the velocity componts can be obtained as

$$u = a\omega\frac{\cosh(k(z + H))}{\sinh(kH)}\cos(kx - \omega t) \tag{6.39}$$

and

$$w = a\omega\frac{\sinh(k(z + H))}{\sinh(kH)}\sin(kx - \omega t) \quad . \tag{6.40}$$

So far the dynamical boundary condition at the surface, (6.27), has not yet been used. Inserting (6.30) and (6.38) into (6.27), we obtain a relation between the angular velocity and the wave number,

$$\omega = \sqrt{gk\tanh(kH)}\,, \tag{6.41}$$

from which, according to (6.7), the phase wave speed can be obtained as a function of the wave number,

$$c = \frac{\omega}{k} = \sqrt{\frac{g}{k}\tanh(kH)} \quad . \tag{6.42}$$

This latter equation is of crucial importance for the understanding of surface wave phenomena. It states that harmonic waves with different wave numbers (or different wave lengths) travel with different speeds. Relations such as (6.41) and (6.42) are called *dispersion relations*.

6.3.1 Some features of linear surface waves

At this point, it is interesting to study the path of a material particle under waves. It should be recalled that the motion of a material particle is always described by (2.1) as a function of the time, t, and the initial position, $\boldsymbol{X}$,

of the particle at $t = t_0$. The particle path can be obtained by solving the differential equation (2.3), which simply defines the velocity of the particle. In component form, this equation can be written as

$$u(\xi, \zeta, t) = \frac{\partial \xi}{\partial t} \ , \quad w(\xi, \zeta, t) = \frac{\partial \eta}{\partial t} \ , \tag{6.43}$$

where ξ and ζ denote the components of the vector valued function $\chi(\boldsymbol{X}, t)$. The problem is complicated by the fact that the velocities in (6.43) have to be evaluated at the actual postion of the material point, $\boldsymbol{x}$. If we assume, however, that the wave motion is weak and thus the particle does not travel far from its initial position, it is legitimate to evalutate the velocities at the intial position,

$$u(X, Z, t) = \frac{\partial \xi}{\partial t} \ , \quad w(X, Z, t) = \frac{\partial \eta}{\partial t} \quad . \tag{6.44}$$

This step simplifies the task of solving a set of two ordinary differential equations, (6.43), to two simple integrations. Inserting the expressions for the velocities (6.39) and (6.40) into (6.44) yields

$$\begin{aligned} \frac{\partial \xi}{\partial t} &= a\omega \frac{\cosh(k(Z+H))}{\sinh(kH)} \cos(kX - \omega t) \\ \frac{\partial \zeta}{\partial t} &= a\omega \frac{\sinh(k(Z+H))}{\sinh(kH)} \sin(kX - \omega t) \quad . \end{aligned} \tag{6.45}$$

Integration in time and elimination of the sin and cos functions by a well-known trigenometric relation finally results in

$$\frac{\xi^2}{\left[a \dfrac{\cosh(k(Z+H))}{\sinh(kH)}\right]^2} + \frac{\zeta^2}{\left[a \dfrac{\sinh(k(Z+H))}{\sinh(kH)}\right]^2} = 1 \ , \tag{6.46}$$

which has the form of an ellipse, see figure Figure 6.1.

6.3.2 Deep-water waves

Linear waves are considered as deep water waves when the water depth is much larger than the wave length:

$$kH \propto \frac{H}{\lambda} \gg 1 \quad . \tag{6.47}$$

Since $\tanh(kH) \to 1$ for $kH \to \infty$, the general dispersion relations (6.41) and (6.42) simplify to

$$\omega = \sqrt{kg} = \sqrt{2\pi \frac{k}{\lambda}} \quad . \tag{6.48}$$

and

$$c = \sqrt{\frac{g}{k}} = \sqrt{\frac{1}{2\pi} g\lambda} \tag{6.49}$$

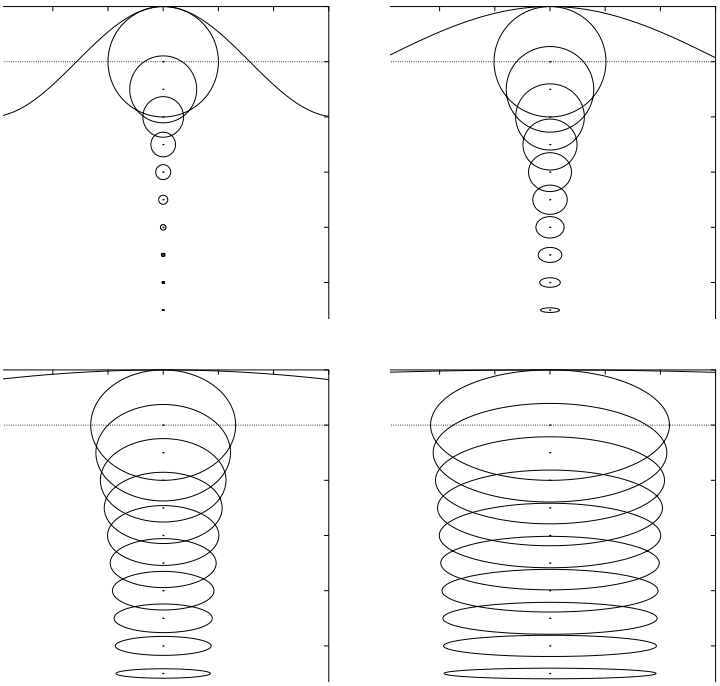

Figure 6.1: Lagrangian particle paths according to linear wave theory with a wave amplitude of $a = 2$ m and a water depth of $H = 10$ m. The wave number is $k = 0.5$ m^{-1} for the upper left, $k = 0.25$ m^{-1} for the upper right, $k = 0.1$ m^{-1} for the lower left, and $k = 0.05$ m^{-1} for the lower right, indicating a shift from relatively deep to relatively shallow water.

Thus, the phase speed and the wave frequency depend on the wave number, k. The longer the waves, the faster the phase velocity. Such waves are called dispersive waves.

6.3.3 Shallow-water approximation

For shallow water, we have to consider the limit $kH \to 0$. In this limit, the Taylor series for tanh shows

$$\begin{aligned} \tanh(x) &= \tanh(0) + x \tanh'(0) + \dots \\ &= \tanh(0) + x(1 - \tanh^2(0)) + \dots \\ &\approx x \quad \text{for } x \to 0 \quad . \end{aligned} \tag{6.50}$$

Thus, for shallow water, the general dispersion relation (6.42) simplifies to

$$c = \sqrt{gH} \, , \tag{6.51}$$

where the phase speed does not any more depend on the wave number, but only on the water depth. Shallow-water thus do not disperse and may keep their form even if they are not sinusoidal.

Chapter 7

Scaling

It has been mentioned in Chapter 5 that geophysical motions are quite accurately described by the Boussinesq-equations, (5.10), because these equations capture the essential physics of density stratified flows without including the often unecessary presence of sound waves. Sound waves are effectively filtered out due to the particular form of the balance of mass, (5.9), for Boussinesq fluids which is formally identical the continuity equation for incompressible fluids.

As discussed in detail in Section 3.7, for a coordinate system turning with the planet, additional transformation terms appear in these equations. Assuming that the acceleration of Earth's center with respect to the inertial system and changes in the angular velocity are negligible, the only two contributions from the motion of the reference system are the Coriolis and the centrifugal acceleration, see (3.66). The latter is always directed outward, perpendicular to the axis of rotation. Its effect leads to a deformation of the spherical shape of the planet, with the distance between the poles being approximately 42 km less than the equatorial diameter. The resultant force on a fluid particle situated on the surface of this body is the vectorial sum of the gravity force and the centrifugal force. It can be shown that this resultant is exactly perpendicular to the surface. Thus, the effect of the Coriolis force can be absorbed by simply redefining the acceleration vector, $\boldsymbol{g}$, as the resultant, and taking $z = x_3$ as the "upward" coordinate, normal to the surface.

With these assumptions, the Boussinesq equations, (5.10), may be written as

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} + 2\epsilon_{ijk}\Omega_j u_k = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} - \frac{g}{\rho_0}\rho\delta_{i3} + \mathcal{F}_i \;, \tag{7.1}$$

where it is understood that all quantities are measured with respect to the rotating system. The exact form of the friction terms, $\mathcal{F}_i$, follows from (5.10).

Due to the presence of gravity, geophysical flows often exhibit a pronounced vertical density stratification. Therefore, it is useful to decompose the density according to

$$\rho = \bar{\rho}(z) + \rho_*(x, y, z, t) \;, \tag{7.2}$$

where $\bar{\rho}(z)$ denotes a static vertically varying component, and $\rho_*(x, y, z, t)$ the perturbation density. In a similar way, also the pressure can be decomposed in to a static and perturbation pressure,

$$p = \bar{p}(z) + p_*(x, y, z, t) \quad . \tag{7.3}$$

The quantities $\bar{\rho}(z)$ and $\bar{p}(z)$ are defined exactly such that in the absence of any motion, the Boussinesq-equations, (7.1), simplify to the hydrostatic balance for motionless fluids,

$$\frac{\partial \bar{p}}{\partial z} = -g\bar{\rho}(z) \quad . \tag{7.4}$$

With the help of the decompositions (7.2) and (7.3), the sum of the pressure gradient and the buoyancy term in (7.1) can be re-written as

$$\begin{aligned} \frac{\partial p}{\partial x_i} + g\rho\delta_{i3} &= \frac{\partial \bar{p}}{\partial x_i} + \frac{\partial p_*}{\partial x_i} + g\bar{\rho}\delta_{i3} + g\rho_*\delta_{i3} \\ &= \left(\frac{\partial \bar{p}}{\partial z}\delta_{i3} + g\bar{\rho}\delta_{i3}\right) + \frac{\partial p_*}{\partial x_i} + g\rho_*\delta_{i3} \\ &= \frac{\partial p_*}{\partial x_i} + g\rho_*\delta_{i3} \ , \end{aligned} \tag{7.5}$$

where in the last step, use was made of the relation, (7.4). With the help of this result, (7.1) can be reformulated as

$$\frac{\mathrm{D}u_i}{\mathrm{D}t} + 2\epsilon_{ijk}\Omega_j u_k = -\frac{1}{\rho_0}\frac{\partial p_*}{\partial x_i} - \frac{g}{\rho_0}\rho_*\delta_{i3} + \mathcal{F}_i \ , \tag{7.6}$$

which has exactly the same form as (7.1), with pressure and density replaced by their perturbations, respectively.

Geophyscal flows are most often turbulent flows, in which it is observed that the velocity can be decomposed according to

$$\boldsymbol{u} = \langle \boldsymbol{u} \rangle + \boldsymbol{u}' \ , \tag{7.7}$$

where $\langle \boldsymbol{u} \rangle$ denotes a deterministic "mean" part, and $\boldsymbol{u}'$ a chaoticly fluctuating "turbulent" part. The same decomposition applies to all other fields, e.g. the pressure and the density. It turns out that the Boussinesq-equations hold in exactly the same form also for the mean quantities, if all variables in (7.6) are replaced by their mean parts. The important difference, however, is that the friction terms. $\mathcal{F}_i$, now include so-called *turbulent stresses* due to the presence of the fluctuations. These turbulent stresses are typically much larger than the viscous stresses discussed in Chapter 4. Physical models for these turbulent stresses are rather complicated and out of the scope of this text. Therefore, we symbolically denote their presence by the vector of the friction forces, $\mathcal{F}_i$.

Now consider a coordinate system located at the planet's surface (see Figure 7.1), with the x-axis pointing towards the East (i.e. into the paper), the y-axis

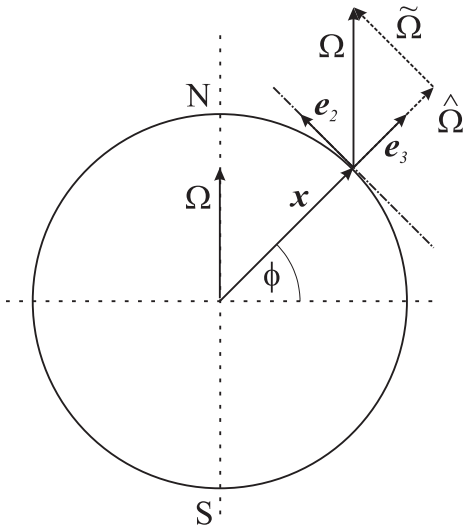

Figure 7.1: Decomposition of Earth's rotation into components directed upward and towards the North.

pointing towards North (i.e. into the direction of $\tilde{\mathbf{\Omega}}$) and the z-axis pointing upward (i.e. into the direction of $\mathbf{\Omega}$. Denoting ϕ as the latitude, the components of $\mathbf{\Omega}$ with respect to this coordinate system are

$$\mathbf{\Omega} = \Omega\cos\phi\,\boldsymbol{e}_2 + \Omega\sin\phi\,\boldsymbol{e}_3\ , \tag{7.8}$$

where we introduced $\Omega = |\mathbf{\Omega}|$ for convenience. The components of the Coriolis term then read

$$\begin{aligned} 2\mathbf{\Omega}\times\boldsymbol{u} &= (2\Omega\cos\phi\,w - 2\Omega\sin\phi\,v) && \boldsymbol{e}_1 \\ &\qquad\qquad +2\Omega\sin\phi\,u && \boldsymbol{e}_2 \\ &\qquad\qquad -2\Omega\cos\phi\,u && \boldsymbol{e}_3 \\ &= (f_*w - fv) && \boldsymbol{e}_1 \\ &\qquad\qquad +fu && \boldsymbol{e}_2 \\ &\qquad\qquad -f_*u && \boldsymbol{e}_3\ , \end{aligned} \tag{7.9}$$

where the symbols $f = 2\Omega\sin\phi$ and $f_* = 2\Omega\cos\phi$ denote the vertical and horizontal *Coriolis parameters*.

With these conventions, we arrive finally at the component form of (7.6),

$$\begin{aligned}
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+f_*w-fv &= -\frac{1}{\rho_0}\frac{\partial p_*}{\partial x}+\mathcal{F}_x \\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu &= -\frac{1}{\rho_0}\frac{\partial p_*}{\partial y}+\mathcal{F}_y \\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}-f_*u &= -\frac{1}{\rho_0}\frac{\partial p_*}{\partial z}-\frac{g}{\rho_0}\rho_*+\mathcal{F}_z \quad .
\end{aligned} \tag{7.10}$$

Evidently, this set of equations is quite complex. To see the relative importance of the different terms, and to find out if some terms can possibly be ignored for some problems, it is necessary to obtain some information about the scales. Let a "typical" length-scale of the motion be denoted by L. As an example, consider waves, for which L would be of the order of the wave length. Next, let us call U the typical velocity change in the x-direction over the length L. Then, the velocity gradient can be written as

$$\frac{\partial u}{\partial x}=\frac{U}{L}\frac{\partial \hat{u}}{\partial \hat{x}}\ , \tag{7.11}$$

where the hat denotes variables that are of the order of one. Calling W the velocity scale of vertical motion, and H the vertical lenght scale, the balance of mass, (5.9) may be re-written in scaled form as

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial y}=\frac{U}{L}\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{W}{H}\frac{\partial \hat{w}}{\partial \hat{z}}=0\ , \tag{7.12}$$

where, for the moment, we study only two-dimensional flows. From (7.12) we immediately see that

$$W=\frac{H}{L}U \quad . \tag{7.13}$$

Since horizontal length scales in geophysical flows are usually very much larger than vertical flows, it is evident that the continuity equation requires that the vertical velocity scale, W, is much smaller than the horizontal velocity scale, U.

If introduce the additional scales T for the time, Ω for the Coriolis parameters at mid-latitudes, P for the dynamic pressure, and F_H for the horizontal friction term, the x-component of (7.10) becomes

$$\begin{aligned}
\frac{U}{T}\frac{\partial \hat{u}}{\partial \hat{t}} &+ \frac{U^2}{L}\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}}+\frac{U^2}{L}\hat{v}\frac{\partial \hat{u}}{\partial \hat{y}}+\frac{U^2}{L}\hat{w}\frac{\partial \hat{u}}{\partial \hat{z}}+\Omega U\frac{H}{L}\hat{f}_*\hat{w}-\Omega U\hat{f}\hat{v} \\
&= -\frac{P}{\rho_0 L}\frac{\partial \hat{p}_*}{\partial \hat{x}}+F_H\hat{\mathcal{F}}_x \quad .
\end{aligned} \tag{7.14}$$

Again, all quantities with a "hat" are of the order of one. Now we divide the

whole equation by the factor ΩU, and obtain

$$\begin{aligned} Ro_T \frac{\partial \hat{u}}{\partial \hat{t}} &+ Ro\left(\hat{u}\frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v}\frac{\partial \hat{u}}{\partial \hat{y}} + \hat{w}\frac{\partial \hat{u}}{\partial \hat{z}}\right) + A\hat{f}_*\hat{w} - \hat{f}\hat{v} \\ &= -\frac{P}{\rho_0 \Omega U L}\frac{\partial \hat{p}_*}{\partial \hat{x}} + E_H \hat{\mathcal{F}}_x \ , \end{aligned} \tag{7.15}$$

where we introduced a couple of dimensionless numbers. The first is the *temporal Rossby number*,

$$Ro_T = \frac{1}{\Omega T} = \frac{\text{local acceleration force}}{\text{Coriolis force}} \ , \tag{7.16}$$

which is small when the time scale of the flow is large compared to the rotation period. The classical *Rossby number*,

$$Ro = \frac{U}{\Omega L} = \frac{\text{intertial force}}{\text{Coriolis force}} \ , \tag{7.17}$$

measures the relative importance of the non-linear advection (or inertial) terms with respect to the vertical Coriolis term. This number is typically small for fast rotation rates. The *horizontal Ekman number*,

$$E_H = \frac{F_H}{\Omega U} = \frac{\text{horizontal friction force}}{\text{Coriolis force}} \ , \tag{7.18}$$

is a number measuring the importance of friction forces compared to rotational forces. In turbulent flows, the friction forces due to turbulence can be quite substantial, but at the moment we cannot quantify them.

The scaling of the pressure-gradient term is not straightforward. One possible choice is a Bernoulli-type scaling of the form $P = \rho_0 U^2$. Under this assumption, the pressure term in (7.15) becomes equal to the Rossby number Ro, and both the advection term and the pressure term are seen to be of the same order, consistent with the derivation of the Bernoulli equation. However, if rotation is important (as implied by our scaling for the time scales), Ro will be small, and the pressure gradient drops out of the problem.

More appropriate for the geophysical problems considered here is the assumption that the horizontal pressure gradient is caused by a lateral tilt of isopycnal surfaces (i.e., surfaces of constant density) resulting from some internal motion of the fluid. If we assume that H denotes the isopycnal tilt over a horizontal distance L, then the lateral pressure gradient scales as $\partial p_*/\partial x \sim P/L = g\Delta\rho H/L$, where $\Delta\rho$ denotes a typical density difference. This scaling follows from dimensional arguments and the physically plausible assumption that both increasing density differences and isopycnal displacements result in an increase of the pressure gradient. This assumption yields $P = g\Delta\rho H$ for the pressure scale. Inserting this scaling for P into (7.15), the scaling factor in front of the pressure gradient term becomes

$$Bo = \frac{g\Delta\rho H}{\rho_0 \Omega U L} = \frac{\text{buoyancy force}}{\text{Coriolis force}} \ , \tag{7.19}$$

which is sometimes referred to as the *buoyancy number*, measuring the relative importance of buoyancy and Coriolis forces.

By very similar arguments, it can be shown that the y-component of (7.10) becomes

$$\begin{aligned} Ro_T \, \frac{\partial \hat{v}}{\partial \hat{t}} \quad &+ \quad Ro \left(\hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} + \hat{w} \frac{\partial \hat{v}}{\partial \hat{z}} \right) + \hat{f} \hat{u} \\ &= \quad -Bo \frac{\partial \hat{p}_*}{\partial \hat{y}} + E_H \hat{\mathcal{F}}_y \quad . \end{aligned} \tag{7.20}$$

Using the scaling for P found above, and assuming, as above, that the density variation ρ_* scales with $\Delta\rho$, the vertical momentum budget in (7.10) can be re-written as

$$\begin{aligned} \frac{UH}{TL} \frac{\partial \hat{w}}{\partial \hat{t}} \quad &+ \quad \frac{U^2 H}{L^2} \hat{u} \frac{\partial \hat{w}}{\partial \hat{x}} + \frac{U^2 H}{L^2} \hat{v} \frac{\partial \hat{w}}{\partial \hat{y}} + \frac{U^2 H}{L^2} \hat{w} \frac{\partial \hat{w}}{\partial \hat{z}} - \Omega U \hat{f}_* \hat{u} \\ &= \quad -\frac{P}{\rho_0 H} \frac{\partial \hat{p}_*}{\partial \hat{z}} - \frac{g \Delta \rho}{\rho_0} \hat{\rho}_* + F_V \hat{\mathcal{F}}_z \end{aligned} \tag{7.21}$$

,

where we additionally assumed that F_V denotes a typcial scale of the vertical friction force.

Dividing as before by ΩU, and multiplying by the *aspect ratio*,

$$A = \frac{H}{L} \tag{7.22}$$

the scaled w-component, (7.21), becomes

$$\begin{aligned} A^2 \left[R_T \, \frac{\partial \hat{w}}{\partial \hat{t}} \quad &+ \quad Ro \, \left(\hat{u} \frac{\partial \hat{w}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{w}}{\partial \hat{y}} + \hat{w} \frac{\partial \hat{w}}{\partial \hat{z}} \right) \right] - A \, \hat{f}_* \hat{u} \\ &= \quad -Bo \, \frac{\partial \hat{p}_*}{\partial \hat{z}} - Bo \, \hat{\rho}_* + A \, E_V \, \hat{\mathcal{F}}_z \, , \end{aligned} \tag{7.23}$$

where, analogously to (7.18), we have introduced the *vertical Ekman number*,

$$E_V = \frac{F_V}{\Omega U} = \frac{\text{vertical friction force}}{\text{Coriolis force}} \quad . \tag{7.24}$$

Now, we consider typical geophysical flows in which the Rossby numbers, the Ekman numbers, and the buoyancy number are all of order 1, i.e. we assume that acceleration, friction, and buoyancy effects are all comparable to the Coriolis force. The additional key assumption that holds in many geophysical situations and greatly simplifies the equations is that the flow can be considered as *shallow*, which is mathematically equivalent to assuming $A \to 0$. The *shallow water assumption* therefore amounts to ignoring all terms multiplied by A or A^2 in (7.15), (7.20), and (7.23). In this context, it is, however, important to note that also the buoyancy number in (7.19) involves the aspect ratio, which would imply

that $Bo \to 0$ for $A \to 0$ if all other parameters remain constant. Here, however, we assume that the decrease in H/L is compensated by a corresponding increase in the density contrast $\Delta\rho$ such that Bo remains finite. Otherwise, $A \to 0$ would eliminate the pressure gradient from the problem, which is not appropriate in most cases.

With these assumptions, and converting back to dimensional variables, (7.15), (7.20), and (7.23) become

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \mathcal{F}_x \ , \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \mathcal{F}_y \ , \qquad (7.25) \\
\frac{\partial p}{\partial z} &= -g\rho \quad .
\end{aligned}$$

These *shallow-water equations* are of fundamental importance in geophysical fluid mechanics. The same equations are valid if the pressure and density are replaced by the perturbation pressure and density.

Chapter 8

Simple Geophysical Applications

8.1 Geostrophic Flows

As the first example, we consider shallow flows with $A \to 0$, and thus start from the shallow water equations, (7.25). Additionally we assume that the rotation is so rapid, that the local and advective accelerations can be ignored, $Ro_T \to 0$, $Ro \to 0$, and that the friction forces are very small compared to the Coriolis force, $E_H \to 0$.

Under these conditions, the shallow water equations, (7.25), reduce to,

$$\begin{aligned} -fv &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ fu &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\,, \end{aligned} \tag{8.1}$$

which is a simple balance between the Coriolis acceleration and the pressure gradient. This is called the *geostrophic balance*. Multiplying the first equation by $-\rho_0 u$, the second by $-\rho_0 v$, and adding the two equations yields

$$u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} = \boldsymbol{u}\cdot\nabla p = 0\,, \tag{8.2}$$

which shows that the flow is not aligned with the pressure gradient, as in non-rotating flows, but *perpendicular* to it. The velocity components of geostrophically balanced flows are

$$\begin{aligned} u &= -\frac{1}{\rho_0 f}\frac{\partial p}{\partial y}\,, \\ v &= \frac{1}{\rho_0 f}\frac{\partial p}{\partial x}\,, \end{aligned} \tag{8.3}$$

a result that is easily obtained by re-arranging (8.1).

The geostrophic balance is the essential component in many geophysical flows. Figure 8.1 shows the atmospheric pressure distribution over the United States for a day in winter 1991. This figure illustrates nicely that the wind direction is with good accuracy aligned with lines of constant pressure, the *isobars*, and thus directed perpendicular to the pressure gradient.

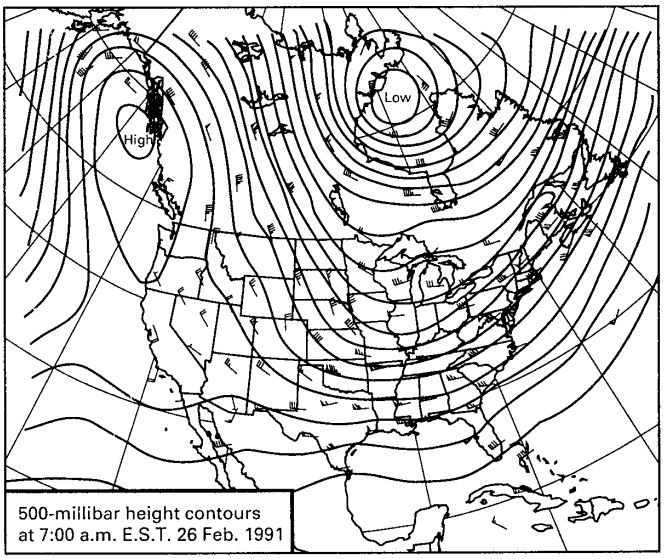

Figure 8.1: Pressure map showing a high and a low pressure cell over the United States with the associated wind stresses at the bottom. The solid lines correspons to isobars, the arrows indicate the wind direction and speed according the meteorological convention.

Since on the Northern Hemisphere, the sign of the Coriolis parameter, f, is positive, it is easy to show from (8.3) that currents flow with the high pressure areas to their right. In consequence, the air rotates clockwise around high-pressure cells and anti-clockwise around low pressure cells. On the Southern Hemisphere, the sense of rotation is exactly opposite. To avoid the distinction between Northern and Southern Hemisphere, meteorologists often use the expressions *cyclonic* and *anti-cyclonic* circulation for low and high pressure cells, respectively.

That geostrophically balanced flows have the high pressure always on their right in the Northern Hemisphere is in agreement with our earlier finding that there the Coriolis acceleration tends to deflect particles to the right, if they move tangentially to the surface of the Earth. In geostrophic flows, this Coriolis acceleration to the right is exactly compensated by the increasing pressure towards the right.

8.2 Ekman Transport

Another interesting special case of (7.25) is obtain by again considering rapidly rotating flows with $Ro_T \to 0$ and $Ro \to 0$. Now, however, we assume that the flow is horizontally homogeneous, implying that there is no horizontal pressure gradient. Under these conditions the Coriolis terms have to be balanced by the friction forces. The resulting equations read

$$\begin{aligned} -fv &= \mathcal{F}_x = \frac{1}{\rho_0}\frac{\partial T_{zx}}{\partial z} \\ fu &= \mathcal{F}_y = \frac{1}{\rho_0}\frac{\partial T_{zy}}{\partial z} \ , \end{aligned} \tag{8.4}$$

where, without losss of generality, the friction terms are identified with the divergence of the stress tensor, see (3.36). Note that due to the assumption of horizontally homogeneous flows, only the vertical part of the divergence of the stress tensor survives. In laminar flows, the components of the stress tensor follow from (4.13), which reduces to

$$\begin{aligned} T_{zx} &= \mu\frac{\partial u}{\partial z} \ , \\ T_{zy} &= \mu\frac{\partial v}{\partial z} \ , \end{aligned} \tag{8.5}$$

for the special case considered here. As remarked above, in turbulent flows, the same equation (8.4) holds, provided all quantities are replaced by their mean values. Then, however, the stress tensor, $\boldsymbol{T}$, includes additional turbulent stresses that are difficult to model.

8.2.1 Integrated transport

No matter what model applies for the stress tensor, equations (8.4) can be vertically integrated. As an example, we integrate from the "bottom" of an infinitely deep ocean to the free surface, ζ. This yields

$$\begin{aligned} U &= \int_{-\infty}^{\zeta} u \, \mathrm{d}z = \frac{1}{\rho_0 f}\int_{-\infty}^{\zeta} \frac{\partial T_{zy}}{\partial z} \, \mathrm{d}z = \frac{1}{\rho_0 f}\Big(T_{zy}(\zeta) - T_{zy}(-\infty)\Big) \\ V &= \int_{-\infty}^{\zeta} v \, \mathrm{d}z = -\frac{1}{\rho_0 f}\int_{-\infty}^{\zeta} \frac{\partial T_{zx}}{\partial z} \, \mathrm{d}z = -\frac{1}{\rho_0 f}\Big(T_{zx}(\zeta) - T_{zx}(-\infty)\Big) \quad . \end{aligned} \tag{8.6}$$

The stresses at $z = -\infty$ are assumed to vanish. The stresses at the free surface correspond to the surface wind stress, $\boldsymbol{\tau}^s$. Thus, (8.6) can be written compactly as

$$\begin{aligned} U &= \frac{\tau_y^s}{\rho_0 f} \ , \\ V &= -\frac{\tau_x^s}{\rho_0 f} \quad . \end{aligned} \tag{8.7}$$

This equation is in form identical to the geostrophic relation (8.3), if the pressure gradient is thought of being replaced by the negative wind stress. Using the same arguments as before, it is easy to show that integrated transport, $\boldsymbol{U}$, is perpendicular to the wind stress. On the Northern Hemisphere it is directed to the right of the wind stress. This is a very surprising result, that has been verified countless times in real-world flows. It has been observed, for example, that icebergs, which have a large vertical extent under water, systematically drift to the right of the wind in the North Atlantic.

8.2.2 The surface Ekman spiral

Now we insert the material law (8.5) into (8.4). Assuming that the viscosity is constant and recalling the the diffusivity of momentum is defined as $\nu = \mu/\rho_0$, we obtain

$$\begin{aligned} -fv &= \nu \frac{\partial^2 u}{\partial z^2} \\ fu &= \nu \frac{\partial^2 v}{\partial z^2} \quad . \end{aligned} \tag{8.8}$$

We will solve these partial differential equations with the boundary conditions

$$\begin{aligned} &T_{zx} = \nu \frac{\partial u}{\partial z} = \frac{\tau_s}{\rho_0}, && \text{for } z = 0, \\ &T_{zy} = \nu \frac{\partial v}{\partial z} = 0, && \text{for } z = 0, \\ &u \to 0, && \text{for } z \to -\infty, \\ &v \to 0, && \text{for } z \to -\infty \quad . \end{aligned} \tag{8.9}$$

These boundary conditions simply state that the velocities and the stresses become very small in great depths, and that the surface stress is aligned with the x-direction.

Now we introduce the complex velocity,

$$V = u + iv \ , \tag{8.10}$$

into (8.8), and obtain

$$\frac{\mathrm{d}^2 V}{\mathrm{d}z^2} = \frac{if}{\nu} V \ , \tag{8.11}$$

for which the solution is

$$V = Ae^{(1+i)z/\delta} + Be^{-(1+i)z/\delta} \quad . \tag{8.12}$$

In this equation, we have introduced the *Ekman depth*,

$$\delta = \sqrt{\frac{2\nu}{f}}. \quad . \tag{8.13}$$

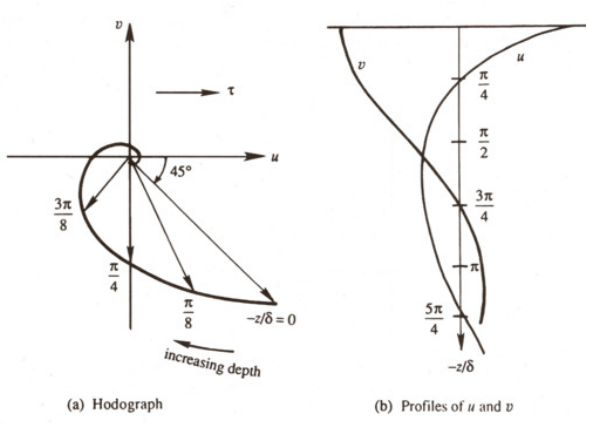

Figure 8.2: Ekman spiral for the ocean surface layer. This figure has been taken from Kundu and Cohen (2008).

The constant B in (8.12) must be zero in order to allow for finite velocity for $z \to -\infty$. After formulating the surface boundary conditions (8.9) for u and v as

$$\nu \frac{\partial V}{\partial z} = \frac{\tau_s}{\rho_0} \quad \text{for } z = 0 \,, \tag{8.14}$$

we obtain

$$A = \frac{\tau_s \delta (1-i)}{2\rho_0 \nu} = \frac{1}{2}\sqrt{2}\frac{\tau_s/\rho_0}{\sqrt{f\nu}}(1-i) = \frac{1}{2}\sqrt{2}\frac{\tau_s/\rho_0}{\sqrt{f\nu}}\sqrt{2}e^{-i\pi/4} \quad . \tag{8.15}$$

Insertion of this into (8.12) gives

$$V = \frac{\tau_s/\rho_0}{\sqrt{f\nu}} e^{z/\delta} e^{i(z/\delta - \pi/4)} \tag{8.16}$$

For u and v we finally obtain

$$u = \frac{\tau_s/\rho_0}{\sqrt{f\nu}} e^{z/\delta} \cos\left(-\frac{z}{\delta} + \frac{\pi}{4}\right), \tag{8.17}$$

$$v = -\frac{\tau_s/\rho_0}{\sqrt{f\nu}} e^{z/\delta} \sin\left(-\frac{z}{\delta} + \frac{\pi}{4}\right). \tag{8.18}$$

An inspection of graphical representations of these velocity profiles shows that the velocity vector describes a spiral with increasing depth, see figure (8.2). This solution, which has been found by the Swedish Oceanographer Vagn Walfrid Ekman, is thus called the Ekman spiral.

Appendix A

Generalized gradients of tensorial quantities

When working with gradients of tensorial quantities, some texts, in particular in continuum mechanics, follow a rationale that is not based on the nabla-operator introduced in Section 1.2. This analysis builds up on the so-called *directional derivative*,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\boldsymbol{T}(\boldsymbol{x}+\alpha\boldsymbol{s})\bigg|_{\alpha=0}\,, \tag{A.1}$$

which describes the change of an arbitrary-order tensor $\boldsymbol{T}$ in the direction of $\boldsymbol{s}$ at position $\boldsymbol{x}$.

A.1 Gradient of a tensor field

The *gradient*, $\operatorname{grad}\boldsymbol{T}$, of a tensor field $\boldsymbol{T}$ is then defined from the implicit relation,

$$\operatorname{grad}\boldsymbol{T}\cdot\boldsymbol{s} \equiv \frac{\mathrm{d}}{\mathrm{d}\alpha}\boldsymbol{T}(\boldsymbol{x}+\alpha\boldsymbol{s})\bigg|_{\alpha=0}\,, \tag{A.2}$$

which is valid for arbitrary coordinate systems (including curvilinear coordinates).

For cartesian coordinates, introducing $\boldsymbol{r}=\boldsymbol{x}+\alpha\boldsymbol{s}$, and using the chain rule for the derivative on the right hand side of (A.2), we find

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\boldsymbol{T}(\boldsymbol{r})\bigg|_{\alpha=0} &= \frac{\partial\boldsymbol{T}}{\partial r_i}\frac{\mathrm{d}r_i}{\mathrm{d}\alpha}\bigg|_{\alpha=0} = \frac{\partial\boldsymbol{T}}{\partial r_i}\bigg|_{\alpha=0} s_i \\ &= \frac{\partial\boldsymbol{T}}{\partial x_i}\boldsymbol{e}_i\cdot\boldsymbol{s} = \left(\frac{\partial\boldsymbol{T}}{\partial x_i}\otimes\boldsymbol{e}_i\right)\cdot\boldsymbol{s}\,,\end{aligned} \tag{A.3}$$

where in the last step we have used (1.18). We conclude from and (A.2) and

(A.3) that the gradient of a tensor $\boldsymbol{T}$ is given by

$$\operatorname{grad} \boldsymbol{T} = \frac{\partial \boldsymbol{T}}{\partial x_i} \otimes \boldsymbol{e}_i \quad . \tag{A.4}$$

From similar arguments, it can be shown that the gradient of a scalar Φ is defined by

$$\operatorname{grad} \Phi = \frac{\partial \Phi}{\partial x_i} \boldsymbol{e}_i \quad . \tag{A.5}$$

A.2 Divergence of vector and tensor fields

The *divergence* of a vector field $\boldsymbol{u}$ corresponds to the trace of the gradient of $\boldsymbol{u}$,

$$\operatorname{div} \boldsymbol{u} \equiv \operatorname{tr}(\operatorname{grad} \boldsymbol{u}) \quad . \tag{A.6}$$

For tensors $\boldsymbol{T}$ of order $n > 1$, the divergence is defined from the recursive implicit relation

$$(\operatorname{div} \boldsymbol{T}) \cdot \boldsymbol{a} \equiv \operatorname{div} (\boldsymbol{T} \cdot \boldsymbol{a}) \, , \tag{A.7}$$

which must hold for any (arbitrary) vector $\boldsymbol{a}$. It should be noted that the divergence of a tensor of order n is a tensor of order $n-1$. Relations (A.6) and (A.7) are valid for arbitrary curvilinear coordinate systems.

For Cartesian coordinates, expression (A.6) for the divergence of a vector field $\boldsymbol{u}$ yields

$$\operatorname{div} \boldsymbol{u} = \frac{\partial u_i}{\partial x_i} \, , \tag{A.8}$$

where we used (1.36). Similarly, making use of (1.18), the definition in (A.7) yields

$$\operatorname{div} \boldsymbol{S} \equiv \frac{\partial S_{ik}}{\partial x_i} \boldsymbol{e}_k \tag{A.9}$$

for the divergence of a second-order tensor $\boldsymbol{S}$. Anlogous expressions for higher-order tensors can be computed from recursively applying (1.18) and (A.9).

Related to the gradient and divergence operators is the *Laplacian* of a scalar field Φ,

$$\nabla^2 \Phi = \operatorname{div} \operatorname{grad} \Phi = \operatorname{div} \left(\frac{\partial \Phi}{\partial x_i} \boldsymbol{e}_i \right) = \frac{\partial^2 \Phi}{\partial {x_i}^2} \quad . \tag{A.10}$$

A.3 Curl of a vector field

The curl of a vector field is defined for arbitrary coordinates system by the relation

$$(\operatorname{rot} \boldsymbol{u}) \cdot \boldsymbol{a} = \operatorname{div} (\boldsymbol{u} \times \boldsymbol{a}) \, , \tag{A.11}$$

where $\boldsymbol{a}$ again denotes an arbitrary vector. Relation (A.11) is valid of all cor-rdinates systems.

For Cartesian coordinates systems, inserting the definition for the cross product in (1.23) yields the explicit expression

$$\operatorname{rot} \boldsymbol{u} = \epsilon_{ijk} \frac{\partial u_j}{\partial x_i} \boldsymbol{e}_k \quad . \tag{A.12}$$

The vector curl may be extended to tensors of order $n > 1$ but since the resulting expressions are rarely used in fluid mechanics their derivation is omitted here.

Bibliography

Kundu, P. K. and I. M. Cohen, 2008: *Fluid Mechanics*. 4th ed., Academic Press, London, UK, 872 pp.

Spurk, J. H., 1989: *Strömungslehre*. 2d ed., Springer-Verlag, Berlin.