Assignment Nr. 2

due 08 November

Problem 1

Consider a straight river of infinite length flowing with a constant velocity, v, in the direction x. At time $t = 0$, a pollutant is released at $x = 0$. For $t > 0$, the concentration of the pollutant, $c(x, t)$, is determined by the advection of the pollutant with the river water and by diffusive spreading. Assume that $c(x, t)$ is given by

$$c(x,t) = t^{-\frac{1}{2}} \, e^{-(x-vt)^2 t^{-1}} \; , \tag{1}$$

which is a simple description of the effects of advection and diffusion.

(a) Plot $c(x, t)$ for $-5 < x < 20$ at times $t = 1$, 3, and 10 with $v = 1$ for the velocity of the river water (non-dimensional variables are used throughout this problem). Briefly explain how the effects of diffusion and advection become evident in your plots.

(b) Consider three swimmers moving with constant absolute velocities w_1, w_2, and w_3 in the river. Assume that the first swimmer moves with $w_1 = v$, and thus passively drifts with the current, the second swims with $w_2 = -v$ against the current, and the third always stays at the same point, $x = x_A$. Assume now that at time $t = t_A$ all swimmers meet at the point $x = x_A$.

   (i) Compute the partial derivatives

$$\frac{\partial c(x,t)}{\partial x} \quad \text{and} \quad \frac{\partial c(x,t)}{\partial t}$$

   for $c(x, t)$ given by (1).

   (ii) Derive expressions for the time rates of concentration change each of the three swimmers “feels” while passing the point $x = x_A$ at time $t = t_A$. (hint: apply the total derivative, using the results derived in part (i).)

   (iii) For $v = 1$, $x_A = 5$, and $t_A = 3$, obtain numerical values for these time rates. Which of the swimmers feels the strongest rate of change in concentration? Which the weakest?

Problem 2

Consider a function, $\boldsymbol{\chi}$, describing the three-dimensional motion of material points in a fluid:

$$\begin{aligned} x_1 &= \chi_1(\boldsymbol{X}, t) = X_1 e^{at} , \\ x_2 &= \chi_2(\boldsymbol{X}, t) = X_2 e^{at} , \\ x_3 &= \chi_3(\boldsymbol{X}, t) = X_3 e^{-2at} , \end{aligned} \tag{2}$$

where a is a constant, t is time, and $\boldsymbol{X}$ the reference position at $t = 0$. All components are given with reference to a fixed orthonormal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$.

(a) Compute the Lagrangian velocity, $\boldsymbol{u}^L(\boldsymbol{X}, t)$, and acceleration, $\boldsymbol{a}^L(\boldsymbol{X}, t)$, from the time derivatives of $\boldsymbol{\chi}$.

(b) Compute the inverse of the equations of motion,

$$\boldsymbol{X} = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t) , \tag{3}$$

from (2), and use the result to eliminate $\boldsymbol{X}$ from $\boldsymbol{u}^L(\boldsymbol{X}, t)$ and $\boldsymbol{a}^L(\boldsymbol{X}, t)$ computed in part (a). Derive expressions for the Eulerian descriptions of the velocity, $\boldsymbol{u}^E(\boldsymbol{x}, t)$, and acceleration, $\boldsymbol{a}^E(\boldsymbol{x}, t)$, respectively.

(c) Compute the acceleration, $\boldsymbol{a}^E(\boldsymbol{x}, t)$, from the material derivative of $\boldsymbol{u}^E(\boldsymbol{x}, t)$. Explicitely show that it corresponds to $\boldsymbol{a}^L(\boldsymbol{X}, t)$ derived in part (b).

(d) Compute $\nabla \cdot \boldsymbol{u}$ and $\nabla \times \boldsymbol{u}$. Is this a potential flow? Why?

Problem 3

Let the velocity field describing the stationary motion of a fluid be given by

$$u_1(\boldsymbol{x}, t) = -a x_2 , \quad u_2(\boldsymbol{x}, t) = a x_1 , \quad u_3(\boldsymbol{x}, t) = 0 , \tag{4}$$

where a is a positive constant, and all components are given with reference to a fixed orthonormal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$.

(a) Show that $\nabla \cdot \boldsymbol{u} = 0$.

(b) Obtain the differential equations defining the *streamlines* for this flow. Show that in this particular case

$$x_1 \mathrm{d}x_1 = -x_2 \mathrm{d}x_2 \quad . \tag{5}$$

Integrate (5) to obtain an implicit equation in x_1 and x_2 determining the streamlines. To which geometrical form do they correspond?

(c) Obtain the differential equations for $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$, defining the *path lines* of the flow. Solve these equations for the path line passing through $\boldsymbol{x} = (1, 0, 0)$ at $t = 0$ (hint: obtain an equation for $\mathrm{d}^2 x_1/\mathrm{d}t^2$, and substitute $\mathrm{d}x_2/\mathrm{d}t$ appearing on the right hand side of this equation.) Do streamlines and path lines coincide in this case? Should they?