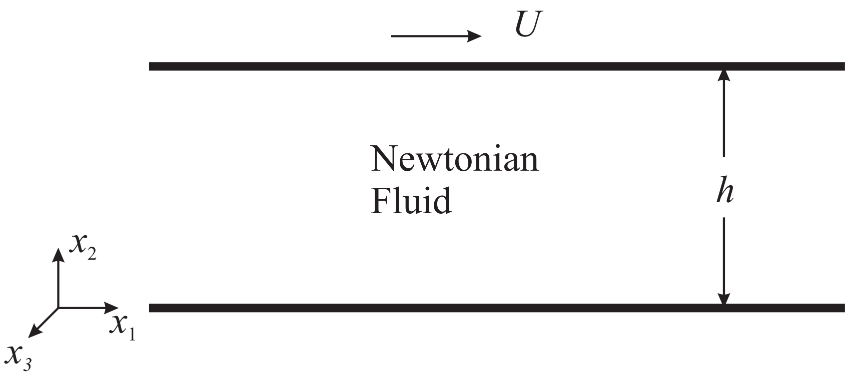
Assignment Nr. 5

due January 17th

Problem 1



Consider the stationary motion of an incompressible Newtonian fluid with constant density, $\rho = \rho_0$, and constant diffusivity of momentum, ν_0, between two parallel, infinite plates of distance h. The upper plate moves parallel to the lower plate with constant velocity, U, in the $\boldsymbol{e}_1$ direction, the lower plate is at rest. The flow is homogeneous in the x_1 and x_3 directions. The only non-zero velocity component is $u_1(x_2)$. The pressure gradient is negligible.

(a) Show that for this flow the Navier-Stokes equations reduce to

$$\nu_0 \frac{\mathrm{d}^2 u_1}{\mathrm{d}x_2^2} = 0 \quad . \tag{1}$$

Discuss, term by term, for what reason individual terms have been neglected in deriving this equation.

(b) Integrate (1) with the help of the boundary conditions $u_1(0) = 0$ and $u_1(h) = U$. Show that the velocity distribution between the plates is linear,

$$u_1 = \frac{U}{h} x_2 \,, \tag{2}$$

and independent of the diffusivity. This is the velocity distribution of the *Couette flow*, which is a famous, prototypical solution of the Navier-Stokes equations (we have encountered this flow already in Problem 1 of the 3rd assignment sheet).

(c) Show that, for the flow considered here, the dissipation function, $\Phi = T_{ij}S_{ij}$, is given by

$$\frac{1}{\rho_0}\Phi = \nu_0 \frac{U^2}{h^2} \ , \tag{3}$$

independent of the spatial position.

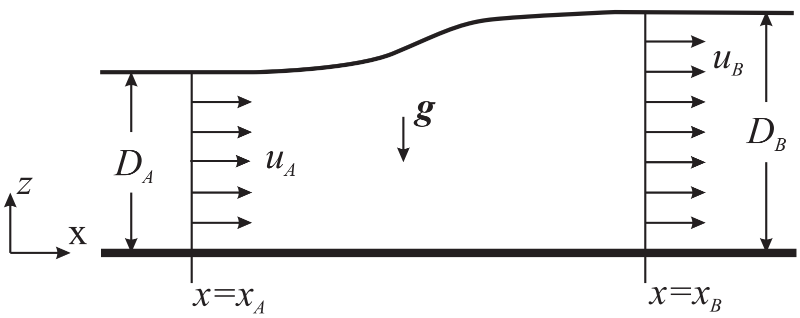
(d) Now assume that the initial temperature distribution is uniform, that there is no heat flux into or out of the plates, and that there is no external energy supply to the fluid. Argue that, under these conditions, the divergence of the heat flux vanishes, and the energy equation can be written as

$$c_v \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{\rho_0}\Phi = \nu_0 \frac{U^2}{h^2} \ , \tag{4}$$

where c_v is heat capacity of the fluid (assumed to be constant here).

(e) For an engineering application of these results, consider a *slide bearing*, where friction between two surfaces moving relative to each other is reduced by pressing lubricating oil in the gap between them. Typical thermodynamical parameters for lubricating oil are $c_v = 2000$ J kg^{-1} K^{-1}, and $\nu_0 = 10^{-5}$ m^2 s^{-1}. Solve (4) for these parameters, and compute the temperature increase of the oil after $t = 60$ s for a gap of $h = 1$ mm and $U = 10$ m s^{-1}.

Problem 2



Consider the stationary flow of an inviscid fluid of constant density in a channel with flat bottom, consisting of two straight sections of different width. At $x = x_A$, located inside the first section, the width of the channel is W_A, the local water depth is D_A, and the velocity, assumed to be vertically homogeneous, is $\boldsymbol{u}_A = u_A \boldsymbol{e}_1$. Further downstream, in the second section, at $x = x_B$, the width of the channel is $W_B < W_A$, the local water depth is D_B, and the velocity, again vertically homogeneous, is $\boldsymbol{u}_B = u_B \boldsymbol{e}_1$. Assume that the pressure at the free surface is equal to the constant ambient pressure, p_0.

(a) Construct a control volume including the planes $x = x_A$ and $x = x_B$, and show that the balance of mass requires that

$$u_A W_A D_A = u_B W_B D_B \quad . \tag{5}$$

(b) Assume that $z = 0$ corresponds to the flat bottom of the channel. Apply the Bernoulli equation between $x = x_A$ and $x = x_B$ along a streamline located at the free surface, and show that

$$\frac{u_A^2}{2} + gD_A = \frac{u_B^2}{2} + gD_B \ , \tag{6}$$

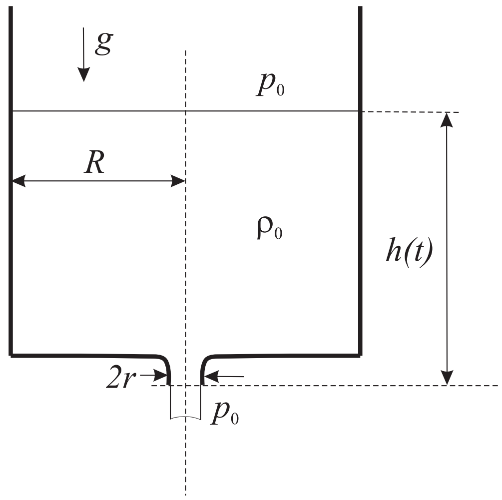
where g is the acceleration of gravity.

(c) Combine (5) and (6) to eliminate u_B from the problem. Show that the resulting equation is a cubic polynomial for D_B of the from

$$2gD_B^3 - \left(2gD_A + u_A^2\right) D_B^2 + \left(\frac{W_A}{W_B} D_A u_A\right)^2 = 0 \quad . \tag{7}$$

(d) Solve (7) either numerically or analytically for D_B. Plot D_B as a function of u_A for $0 \leq u_A \leq 2$ m s^{-1}, using the parameters $g = 9.81$ m s^{-2}, $D_A = 5$ m, and $W_A/W_B = 2$. (Hint: better don't try this without a mathematical software like Mathematica or Maple. Note that there are three possibly different roots of (7). Only one yields physically reasonable results.)

Problem 3



Consider a cylindrical vessel of radius R, filled with an inviscid fluid of constant density, ρ_0. At the bottom of the vessel is a rounded cylindrical orifice of radius r, through which the water leaves the vessel in form of a non-contracting jet with pressure equal to the ambient pressure, p_0.

(a) Construct a control volume and show that the balance of mass requires that the change of the water level, h, is given by

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\left(\frac{r}{R}\right)^2 u_B \ , \tag{8}$$

where u_B denotes the velocity of the jet leaving the orifice.

(b) Assume that the flow is quasi-stationary and $r/R \ll 1$. By applying the Bernoulli equation along a streameline between the water surface and the bottom of the orifice, show that the velocity of the jet leaving the orifice is given by

$$u_B = \sqrt{2gh} \ , \tag{9}$$

where g is the acceleration of gravity. This is the famous *Toricelli* formula. Inserting (9) into (8), show that change of the water level is governed by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t} = Ch^{\frac{1}{2}} \ , \tag{10}$$

where $C = -\sqrt{2g}(r/R)^2$ is constant.

(c) Solve (10), and show that the evolution of the water level is given by

$$h(t) = \left(h_0^{\frac{1}{2}} + \frac{C}{2}(t - t_0) \right)^2 \ , \tag{11}$$

where $h = h_0$ is the initial water level at $t = t_0$.

(d) For $g = 9.81$ m s^{-2}, $R = 0.5$ m, and $r = 0.01$ m, compute the time required for the water level to descend from $h_0 = 0.5$ m to $h_1 = 0.2$ m. Plot the evolution of $h(t)$ for this range of levels.