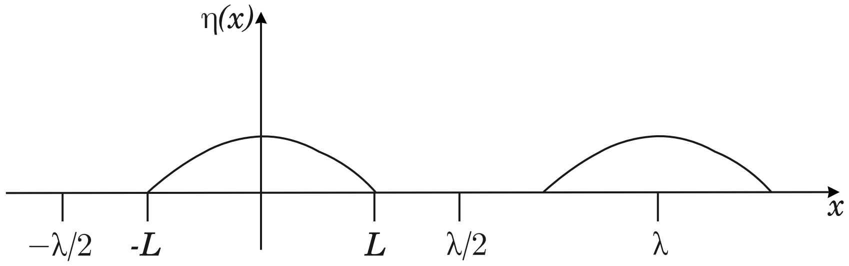
Assignment Nr. 6

due January 31st

Problem 1



Let's consider the motion of surface gravity waves in a horizontally infinite fluid with constant depth, H. Let's further assume that at time $t = t_0$, the free surface displacement η corresponds to a periodic extension of the form:

$$\eta_0(x) = \begin{cases} 0 & \text{if } -\lambda/2 \leq x < -L \\ \cos(\frac{\pi}{2L}x) & \text{if } -L \leq x < L \\ 0 & \text{if } L \leq x < \lambda/2 \quad . \end{cases} , \tag{1}$$

where λ corresponds to the distance between the individual "bumps" (see figure). As any other symmetric and periodic function, $\eta_0(x)$ can be expanded in terms of a Fourier series of the form

$$\eta_0(x; N) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos k_n x \ , \tag{2}$$

which converges to $\eta_0(x)$ for $N \to \infty$. Here, $k_n = 2\pi n/\lambda$ denotes the wave number of the nth Fourier mode with coefficient a_n. The Fourier coefficients for the function described above are given by

$$a_0 = \frac{8L}{\pi\lambda} \ , \quad a_n = \frac{8L\lambda\cos(2\pi nL/\lambda)}{\pi(\lambda^2 - 16L^2n^2)} \quad \text{for } n \neq \frac{\lambda}{4L} \quad . \tag{3}$$

Now let the time evolution of the periodic surface displacement be goverened by the linear equations for surface gravity waves. Since each of the Fourier modes is a solution of these equations, η evolves according to

$$\eta(x, t; N) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos(k_n x - \omega_n t) \ , \tag{4}$$

where ω_n denotes the circular frequency of mode n.

(a) For $L = 1$ m, $\lambda = 501$ m, a water depth of $H = 5000$ m, and $g = 9.81$ m s^{-2} for the acceleration of gravity, use the dispersion relation for surface gravity waves to plot ω_n and the phase velocity, c_n, as functions of the wave number, k_n, for $1 \leq n \leq 1000$. What does it mean that different Fourier components of η move with different phase speeds?

(b) Plot $\eta(x, t; N)$ from (4) using the parameters given above for the range $0 \leq x \leq 100$ m at times $t = 0, 20$ s. Use $N = 1000$ Fourier modes. (You may program the solution of this little problem in any programming language, or use any mathemactical software tool).

Problem 2

Consider again the wave problem described in problem 2, but now with the parameters $a = 100$ km, $\lambda = 1001$ km, and $H = 5000$ m. This corresponds to a simple description of the surface displacement caused by a Tsunami resulting from an earthquake.

(a) Plot the phase velocity, c_n, as a function of the wave length for the 10 Fourier components with longest waves lengths. How long do these components approximately need to cross the Pacific Ocean?

(b) Plot $\eta(x, t; N)$ from (4) using these parameters for the range $0 \leq x \leq 2000$ km at time $t = 1$ hour. Use $N = 1000$ Fourier modes.

(c) Repeat (b), but now using the dispersion relation $\omega_n(k_n)$ in the long-wave (or shallow-water) limit. Since the length of the disturbance, $2L$, is much larger than the water depth, this should lead to a good approximation. What differences do you see between the results for the long-wave approximation and the full dispersion relation? Can you explain them?