Assignment Nr. 1

due 27 April

Problem 1

Verify the identities

(a) $A_{iykzd}\delta_{kl} = A_{iylzd}$ ('exchange property') ,

(b) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$,

(c) $\varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$,

(d) $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ ('ε-δ identity') ,

where δ_{ij} denotes the *Kronecker-δ* and ε_{ijk} the *alternating symbol*. Hint: (b) and (c) follow from (d).

Problem 2

Let A_{ij} be a symmetric and B_{ij} a skew-symmetric second-order tensor. Show that the *contraction* or the *inner product* of a symmetric and an anti-symmetric second-order tensor is always zero, i.e. $A_{ij}B_{ij} = 0$.

Problem 3

Use indical notation to verify the following identities:

(a) $\nabla \times \nabla c = \mathbf{0}$

(b) $(d\nabla c) \times \nabla c = \mathbf{0}$

(c) $\nabla \cdot (\nabla \times \boldsymbol{a}) = \mathbf{0}$

(d) $\boldsymbol{a} \cdot \nabla \boldsymbol{a} = (\nabla \times \boldsymbol{a}) \times \boldsymbol{a} + \frac{1}{2}\nabla(\boldsymbol{a} \cdot \boldsymbol{a})$

(e) $\nabla \times \nabla^2 \boldsymbol{a} = \nabla^2(\nabla \times \boldsymbol{a})$

(f) $\nabla \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}(\nabla \cdot \boldsymbol{b}) + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - \boldsymbol{b}(\nabla \cdot \boldsymbol{a}) - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b}$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, and c and d scalars. Hint: for parts (d) and (f), use the ε-δ identity. For each case, indicate if the equalities hold for a scalar, a vector, or a second-order tensor.

Problem 4

For arbitrary ω_k, let the components of a tensor $\boldsymbol{W}$ be given by

$$W_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k \quad . \tag{1}$$

(a) Show that $\boldsymbol{W}$ is a skew-symmetric tensor.

(b) Using the relation $\varepsilon_{ijk}\varepsilon_{ijl} = 2\delta_{kl}$ (see above), show that (1) can be inverted to yield

$$\omega_i = \varepsilon_{ijk}W_{kj} \quad . \tag{2}$$

Problem 5

With U being a random variable with probability density function $f(V)$, and a being a constant, use the definition of the mean to show that

- $\langle a \rangle = a$
- $\langle aU \rangle = a\langle U \rangle$
- $\langle\langle U \rangle\rangle = \langle U \rangle$
- $\langle u \rangle = 0$, where $u = U - \langle U \rangle$

(Hint: Check the appendix of the lecture notes, where the basics of probability theory are briefly discussed.)