Assignment Nr. 5

due 22 June

Problem 1

Consider the transport equation for the instantaneous temperature, θ, in a Boussinesq-fluid without radiative sources, as discussed in the lecutre notes:

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \nu_\theta \frac{\partial^2 \theta}{\partial x_i \partial x_i} \,, \tag{1}$$

where ν_θ denotes the molecular diffusivity of heat. Starting from (1), show, term by term, that the transport equation for the variance of the temperature fluctuations can be written as

$$\frac{\partial \langle \theta'^2 \rangle}{\partial t} = D_\theta + P_\theta - \chi_\theta \,, \tag{2}$$

where we introduced the transport term

$$D_\theta = -\frac{\partial}{\partial x_i} \left(\overline{u}_i \langle \theta'^2 \rangle + \langle u_i' \theta'^2 \rangle - \nu_\theta \frac{\partial \langle \theta'^2 \rangle}{\partial x_i} \right) \,, \tag{3}$$

the variance production term,

$$P_\theta = -2 \langle u_i' \theta' \rangle \frac{\partial \overline{\theta}}{\partial x_i} \,, \tag{4}$$

and the molecular mixing rate (or molecular destruction rate) of temperature variance,

$$\chi_\theta = 2\nu_\theta \left\langle \frac{\partial \theta'}{\partial x_i} \frac{\partial \theta'}{\partial x_i} \right\rangle \quad . \tag{5}$$

Explain the physical meaning of all terms appearing in (2)–(5). Identify terms that are positive/negative definite, and thus serve as source/sink terms under arbitrary conditions.

Problem 2

In real-ocean flows, the vertical turbulent heat flux, $\langle w'\theta' \rangle$, is notoriously difficult to estimate from the velocity-temperature covariance. The main problem is that temperature and velocity fluctuations are almost exclusively determined by non-turbulent processes like internal waves. A frequently-used alternative approach is the *Osborn-Cox model* that starts by expressing the turbulent heat flux with the help of a gradient law of the form

$$\langle w'\theta' \rangle = -K_\theta \frac{\partial \overline{\theta}}{\partial z}, \tag{6}$$

where K_θ is the *turbulent diffusivity* of heat. It is further assumed that turbulence is stationary, that all transport terms can be ignored, and that the mean temperature gradient is purely vertical.

Starting from (2), show that under these conditions, the turbulent diffusivity in (6) can be rewritten as

$$K_\theta = C\nu_\theta \ , \tag{7}$$

where we introduced the *Cox number*,

$$C = \frac{\left\langle \dfrac{\partial \theta'}{\partial x_i} \dfrac{\partial \theta'}{\partial x_i} \right\rangle}{\left(\partial \overline{\theta} / \partial z\right)^2} \quad . \tag{8}$$

According to (7), the Cox number describes the increase of the effective diffusivity in a turbulent flow compared to purely molecular heat conduction. In a high-Re turbulent flow, $C \gg 1$ because the small-scale turbulent temperature gradients are much larger than the mean vertical gradient (this is similar to our coffee with milk from Chapter 1 after stirring). The Osborn-Cox model requires reliable measurements of small-scale temperature gradients, which can be obtained with the help of specialized *temperature microstructure* sensors.

Problem 3

For isotropic turbulence, the components of *any* Reynolds-averaged fourth-order tensor can be expressed as the linear combination of terms of the form $\delta_{ij}\delta_{kl}$, where it should be noted that only three indepent permutations of indices exist. One example for such a fourth-order isotropic relation is the following:

$$\left\langle \frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_l} \right\rangle = \alpha \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk} \ , \tag{9}$$

where α, β, and γ are arbitrary linear factors.

(a) Show that for incompressible flows ($\partial u_i' / \partial x_i = 0$) the linear factors are related according to

$$3\alpha + \beta + \gamma = 0 \quad . \tag{10}$$

(b) Show that for homogenous, incompressible turbulence

$$\left\langle \frac{\partial u_i'}{\partial x_j} \frac{\partial u_j'}{\partial x_l} \right\rangle = 0 \quad . \tag{11}$$

Hint: Start from $(\partial / \partial x_j)\left\langle u_i' \partial u_j' / \partial x_l \right\rangle = 0$, which is obviously valid for homogenous turbulence.

(c) Use (11) to conclude from (9) that

$$\alpha + \beta + 3\gamma = 0 \quad . \tag{12}$$

(d) Use (10) and (12) to show that for homogenous incompressible flows (9) reduces to

$$\left\langle \frac{\partial u_i'}{\partial x_j} \frac{\partial u_k'}{\partial x_l} \right\rangle = \beta(\delta_{ik}\delta_{jl} - \frac{1}{4}\delta_{ij}\delta_{kl} - \frac{1}{4}\delta_{il}\delta_{jk}) \quad . \tag{13}$$

(e) Starting from the definition of the dissipation rate (for homogenous turbulence),

$$\varepsilon = \nu \left\langle \frac{\partial u_i'}{\partial x_j} \frac{\partial u_i'}{\partial x_j} \right\rangle , \tag{14}$$

use (13) to show the $\varepsilon = (15/2)\nu\beta$.

(f) Show from (13) that $\beta = \langle(\partial u_1'/x_3)^2\rangle$. Hence, show that in isotropic turbulence the dissipation rate is given by

$$\varepsilon = \frac{15}{2}\nu \left\langle \left(\frac{\partial u_1'}{\partial x_3} \right)^2 \right\rangle \quad . \tag{15}$$

This important relation is essential to compute ε from so-called shear-microstructure profilers, measuring the component of the shear perpendicular to their profiling direction. It is only valid if turbulence is isotropic at the smallest (dissipative) scales, which is, however, always the case if the Reynolds number is very large.