A detail derivation of equation (14) in Burchard and Hofmeister (2008).

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\[
\frac{\partial \phi}{\partial t} = \frac{1}{D} \int_{-H}^{H} g z (\tilde{\rho} - \rho) \, dz \quad (1)
\]

\[
\frac{\partial \phi}{\partial t} = \frac{g}{D} \int_{-H}^{H} z (\tilde{\rho} - \rho) \, dz + \frac{g}{D} \left( \frac{1}{D} \right) \int_{-H}^{H} z (\tilde{\rho} - \rho) \, dz
\]

\[
= \frac{g}{D} \int_{-H}^{H} z \frac{\partial (\tilde{\rho} - \rho)}{\partial t} \, dz + \frac{g}{D} \frac{\partial}{\partial t} (\tilde{\rho} - \rho) + \frac{g}{D} \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
= \frac{g}{D} \int_{-H}^{H} z \frac{\partial (\tilde{\rho} - \rho)}{\partial t} \, dz - \frac{g}{D} \eta \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
= \frac{g}{D} \frac{\partial}{\partial t} (\eta - H) - \frac{g}{D} \int_{-H}^{H} z \frac{\partial (\tilde{\rho} - \rho)}{\partial t} \, dz + \frac{g}{D} \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
= \frac{g}{2} (\eta - H) \left[ \frac{1}{D} \int_{-H}^{H} z \rho \, dz + \frac{\partial}{\partial t} \left( \frac{1}{D} \right) \int_{-H}^{H} z \rho \, dz \right] - \frac{g}{D} \int_{-H}^{H} z \frac{\partial (\tilde{\rho} - \rho)}{\partial t} \, dz - \frac{g}{D} \eta \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
\quad \quad + \frac{g}{D} \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
= \frac{g}{2D} (\eta - H) \left( \int_{-H}^{H} \frac{\partial}{\partial t} z \rho \, dz + \frac{\partial}{\partial t} \int_{-H}^{H} z \rho \, dz \right) - \frac{g}{2D} (\eta - H) \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
- \frac{g}{D} \eta \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz + \frac{g}{D} \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
= \frac{g}{2D} (\eta - H) \int_{-H}^{H} \frac{\partial}{\partial t} z \rho \, dz - \frac{g}{D} \int_{-H}^{H} z \frac{\partial}{\partial t} \rho \, dz - \frac{g}{2D} (\eta - H) \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
+ \frac{g}{2D} (\eta - H) \rho \frac{\partial}{\partial t} \eta \frac{\partial}{\partial t} + \frac{g}{D} \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
= \frac{g}{2D} (\eta - H) \int_{-H}^{H} \frac{\partial}{\partial t} z \rho \, dz - \frac{g}{D} \int_{-H}^{H} z \frac{\partial}{\partial t} \rho \, dz - \frac{g}{2D} (\eta - H) \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
+ \frac{g}{2D} (\eta - H) (\tilde{\rho} + \rho) \frac{\partial}{\partial t} \eta \frac{\partial}{\partial t} + \frac{g}{D} \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

I.E. \( \frac{\partial \phi}{\partial t} = \frac{g}{2D} (\eta - H) \int_{-H}^{H} \frac{\partial}{\partial t} z \rho \, dz - \frac{g}{D} \int_{-H}^{H} z \frac{\partial}{\partial t} \rho \, dz + \frac{g}{2D} (\eta - H) \tilde{\rho} \frac{\partial}{\partial t} \eta \frac{\partial}{\partial t} - \frac{g}{D} \eta \tilde{\rho} \frac{\partial}{\partial t} \eta \frac{\partial}{\partial t} + \frac{g}{D} \frac{\partial}{\partial t} \int_{-H}^{H} z \tilde{\rho} \, dz
\]

\[
\frac{\partial \rho}{\partial t} = -u \nabla_{h} \rho - w \frac{\partial}{\partial z} \rho + \frac{k_{v}}{z} \frac{\partial}{\partial z} \rho + \nabla_{h} k_{h} \nabla_{h} \rho + Q \quad (3)
\]

Substitute the first and second terms of \( \frac{\partial \rho}{\partial t} \) in RHS of eq. (2) with eq. (3).

- First, we consider the source term for density, i.e.,
\[ \frac{g}{2D} (\eta - H) \int_{-H}^{\eta} Q dz - \frac{g}{D} \int_{-H}^{\eta} zQ dz \]

\( \eta - H \) is irrelevant with \( z \), we have:

\[ \frac{g}{2D} (\eta - H) \int_{-H}^{\eta} Q dz - \frac{g}{D} \int_{-H}^{\eta} zQ dz = \frac{g}{D} \int_{-H}^{\eta} (\eta - \frac{H}{2} - z) Q dz = \frac{g}{D} \int_{-H}^{\eta} (\eta - \frac{D}{2} - z) Q dz \quad (4) \]

Now we have got the term (G) in equation (14) of Burchard and Hofmeister (2008), hereinafter B & H (2008). Similarly, we can also have the term (H) for the horizontal diffusion term \( \nabla_h k \nabla_h \rho \).

➢ For the vertical diffusion term, we have:

\[ \frac{g}{2D} (\eta - H) \int_{-H}^{\eta} \frac{\partial}{\partial z} k_v \frac{\partial \rho}{\partial z} dz - \frac{g}{D} \int_{-H}^{\eta} \frac{\partial}{\partial z} k_v \frac{\partial \rho}{\partial z} dz \]

\[ = \frac{g}{2D} (\eta - H) k_v \int_{-H}^{\eta} \frac{\partial \rho}{\partial z} dz - \frac{g}{D} k_v \int_{-H}^{\eta} \frac{\partial \rho}{\partial z} dz + \frac{g}{D} \int_{-H}^{\eta} k_v \frac{\partial \rho}{\partial z} dz \]

\[ = \frac{g}{2D} k_v \left( \eta \frac{\partial \rho}{\partial z} - \frac{\partial \rho}{\partial z} - H \frac{\partial \rho}{\partial z} + H \frac{\partial \rho}{\partial z} \right) - \frac{g}{D} k_v \left( \eta \frac{\partial \rho}{\partial z} + H \frac{\partial \rho}{\partial z} \right) \]

\[ + \frac{g}{D} \int_{-H}^{\eta} k_v \frac{\partial \rho}{\partial z} dz \]

\[ = - \frac{g}{2D} k_v \left( \eta \frac{\partial \rho}{\partial z} + H \frac{\partial \rho}{\partial z} \right) + \frac{\rho_0}{D} \int_{-H}^{\eta} P_b dz \]

\[ = - \frac{\rho_0}{2} \left( P_b^e + P_b^h \right) + \frac{\rho_0}{D} \int_{-H}^{\eta} P_b dz \quad (5) \]

Now we have got the terms (E) and (F) in equation (14) of B & H (2008).

Before further derivation, note that we have following equations:

\[ \rho = \bar{\rho} + \bar{\rho} , \quad u = \bar{u} + \bar{u} , \quad w = \bar{w} + \bar{w} , \quad \int_{-H}^{\eta} \bar{\rho} dz = 0 , \quad \int_{-H}^{\eta} \bar{u} dz = 0 \]

\[ w \frac{\partial \rho}{\partial z} = (\bar{w} + \bar{w}) \frac{\partial (\bar{\rho} + \bar{\rho})}{\partial z} = \bar{w} \frac{\partial \bar{\rho}}{\partial z} + \bar{w} \frac{\partial \bar{\rho}}{\partial z} \quad (6) \]

\[ u \nabla_h \rho = \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} \quad (7) \]

The rest of the first and second terms in RHS of eq. (2) after combining eq. (3) are:

\[ - \frac{g}{2D} (\eta - H) \int_{-H}^{\eta} (u \nabla_h \rho + w \frac{\partial \rho}{\partial z}) dz + \frac{g}{D} \int_{-H}^{\eta} z(u \nabla_h \rho + w \frac{\partial \rho}{\partial z}) dz \quad (8) \]

➢ Substitute \( u \nabla_h \rho \) and \( w \frac{\partial \rho}{\partial z} \) in eq. (8) with eqs. (6) and (7), we can have the terms (C) and (D) in equation (14) of B & H (2008), then the rest of eq. (8) would be:

\[ - \frac{g}{2D} (\eta - H) \int_{-H}^{\eta} (\bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{w} \frac{\partial \bar{\rho}}{\partial z}) dz + \frac{g}{D} \int_{-H}^{\eta} z(\bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{w} \frac{\partial \bar{\rho}}{\partial z}) dz \quad (9) \]
First for the horizontal advection term in eq. (9):

\[-\frac{g}{2D} (\eta - H) \int_{-H}^{\eta} (\bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho}) dz + \frac{g}{D} \int_{-H}^{\eta} z(\bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho}) dz\]

\[= - \frac{g}{2D} (\eta - H) \int_{-H}^{\eta} 1dz + \bar{u} \int_{-H}^{\eta} \nabla_h \bar{\rho} dz + \bar{u} \int_{-H}^{\eta} \bar{u} dz\]

\[+ \frac{g}{D} \left( \bar{u} \nabla_h \bar{\rho} \int_{-H}^{\eta} zdz + \bar{u} \int_{-H}^{\eta} z \nabla_h \bar{\rho} dz + \nabla_h \bar{\rho} \int_{-H}^{\eta} z \bar{u} dz \right)\]

\[= - \frac{g}{2D} (\eta - H) \bar{u} \int_{-H}^{\eta} \nabla_h \bar{\rho} dz + \frac{g}{D} \left( \bar{u} \int_{-H}^{\eta} z \nabla_h \bar{\rho} dz + \nabla_h \bar{\rho} \int_{-H}^{\eta} z \bar{u} dz \right)\]

\[= - \frac{g}{2D} (\eta - H) \bar{u} \left( \nabla_h \bar{\rho} - \bar{\rho}_h \nabla_h \eta - \bar{\rho}_h \nabla_h H \right)\]

\[+ \frac{g}{D} \left( \bar{u} \int_{-H}^{\eta} z \nabla_h \bar{\rho} dz + \nabla_h \bar{\rho} \int_{-H}^{\eta} z \bar{u} dz \right)\]

I.E., \(-\frac{g}{2D} (\eta - H) \int_{-H}^{\eta} (\bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho}) dz + \frac{g}{D} \int_{-H}^{\eta} z(\bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho} + \bar{u} \nabla_h \bar{\rho}) dz\]

\[= \frac{g}{2D} (\eta - H) \bar{u} \left( \nabla_h \eta + \bar{\rho}_h \nabla_h H \right)\]

\[+ \frac{g}{D} \left( \bar{u} \int_{-H}^{\eta} z \nabla_h \bar{\rho} dz + \nabla_h \bar{\rho} \int_{-H}^{\eta} z \bar{u} dz \right)\]  

(8)

Second, for the vertical advection term in eq. (9):

\[-\frac{g}{2D} (\eta - H) \int_{-H}^{\eta} \frac{\partial \bar{\rho}}{\partial z} dz + \frac{g}{D} \int_{-H}^{\eta} z \frac{\partial \bar{\rho}}{\partial z} dz\]

\[= - \frac{g}{2D} (\eta - H) \left( \int_{-H}^{\eta} \frac{\partial \bar{w} \bar{\rho}}{\partial z} dz - \int_{-H}^{\eta} \bar{\rho} \frac{\partial \bar{w}}{\partial z} dz \right) + \frac{g}{D} \int_{-H}^{\eta} z \left( \frac{\partial \bar{w} \bar{\rho}}{\partial z} - \bar{\rho} \frac{\partial \bar{w}}{\partial z} \right) dz\]

\[= - \frac{g}{2D} (\eta - H) (\bar{w} \bar{\rho}) \int_{-H}^{\eta} z \bar{\rho} dz - \frac{g}{D} \int_{-H}^{\eta} \bar{w} \bar{\rho} dz + \frac{g}{D} \int_{-H}^{\eta} z \bar{\rho} \frac{\partial \bar{w}}{\partial z} dz\]  

(9)

Note that in deriving eq. (9) \(\int_{-H}^{\eta} \bar{\rho} \frac{\partial \bar{w}}{\partial z} dz = 0\).

Combining \(\bar{w} = -\bar{u} \nabla_h \eta^{\eta H} + \left( \frac{\partial \bar{\eta}}{\partial \zeta} + \bar{u} \nabla_h \eta \right) z \frac{H^{\eta H}}{D} \) with eq. (9), we have:

\[-\frac{g}{2D} (\eta - H) \int_{-H}^{\eta} \bar{w} \frac{\partial \bar{\rho}}{\partial z} dz + \frac{g}{D} \int_{-H}^{\eta} z \bar{w} \frac{\partial \bar{\rho}}{\partial z} dz\]

\[= - \frac{g}{2D} (\eta - H) \left( \left( \frac{\partial \bar{\eta}}{\partial \zeta} + \bar{u} \nabla_h \eta \right) \bar{\rho}_h + \bar{u} \nabla_h \eta \bar{\rho}_h \right) + \frac{g}{D} \frac{\partial \bar{\eta}}{\partial \zeta} \bar{\rho}_h + \frac{g}{D} \frac{\partial \bar{\eta}}{\partial \zeta} \bar{u} \nabla_h \eta\]  

(12)

\[= - \frac{g}{D} \int_{-H}^{\eta} \bar{u} \nabla_h \eta - \frac{g}{D} \int_{-H}^{\eta} \bar{w} \bar{\rho} dz + \frac{g}{D} \int_{-H}^{\eta} z \bar{\rho} dz \]  

(13)

Finally, combining the horizontal and vertical advection terms (eq.(8) and eq. (10)) and \(\frac{g}{2D} (\eta - H) \bar{\rho}_h \frac{\partial \bar{\eta}}{\partial \zeta} - \frac{g}{D} \frac{\partial \bar{\rho}_h}{\partial \zeta} + \frac{g}{D} \frac{\partial \bar{\eta}}{\partial \zeta} \int_{-H}^{\eta} \bar{w} \bar{\rho} dz\) in eq. (2), we have the following equation:
\[
\frac{g}{2D} (\eta - H) \frac{\partial \eta}{\partial t} - \frac{g}{D} \eta \frac{\partial \bar{\rho}}{\partial t} + \frac{g}{D} \frac{\partial \eta}{\partial t} \int_{-H}^{\eta} z \bar{\rho} \, dz + \frac{g}{2D} (\eta - H) \bar{u} (\bar{\rho} \nabla \eta + \bar{\rho} \nabla H)
\]
\[
\frac{g}{D} \bar{u} \int_{-H}^{\eta} z \nabla \bar{\rho} \, dz + \frac{g}{D} \nabla \bar{\rho} \int_{-H}^{\eta} z \bar{u} \, dz \quad \text{(A1) term (B) in eq. (14) of B & H (2008)}
\]
\[
-\frac{g}{2D} (\eta - H) \left[ \left( \frac{\partial \bar{\rho}}{\partial t} + \bar{u} \nabla \bar{\rho} \right) \bar{\rho} + \bar{u} \nabla H \bar{\rho} \right] + \frac{g}{D} \eta \frac{\partial \bar{\rho}}{\partial t} \bar{\rho} + \frac{g}{D} \eta \frac{\partial \bar{\rho}}{\partial t} \bar{u} \nabla \eta \quad \text{(A2)}
\]
\[
-\frac{g}{D} H \bar{\rho} \bar{\rho} \nabla H \bar{\rho} \int_{-H}^{\eta} \bar{w} \bar{\rho} \, dz + \frac{g}{D} \int_{-H}^{\eta} z \bar{\rho} \, dz \nabla \bar{\rho} \quad \text{(A3)}
\]
\[
\frac{g}{D} \int_{-H}^{\eta} \bar{w} \bar{\rho} \, dz = \frac{g}{D} \nabla \bar{\rho} \int_{-H}^{\eta} z \bar{\rho} \, dz \quad \text{(A4)}
\]

(11)

And note that:

\[
-\frac{g}{D} \int_{-H}^{\eta} \bar{w} \bar{\rho} \, dz = \frac{g}{D} \nabla \bar{\rho} \int_{-H}^{\eta} z \bar{\rho} \, dz \quad \text{(**)}
\]

In order to derive the above equation, substitute \( \bar{w} \) with \( \bar{w} = -\bar{u} \nabla H \frac{\partial \eta}{\partial t} + \eta \nabla h + H \),

\[
\frac{g}{D} \int_{-H}^{\eta} \bar{w} \bar{\rho} \, dz = -\frac{g}{D} \int_{-H}^{\eta} \left( \bar{u} \cdot \nabla H + \frac{\partial \eta}{\partial t} + \bar{u} \cdot \nabla \eta \right) \frac{\partial \eta}{\partial t} \bar{\rho} \, dz = -\frac{g}{D} \int_{-H}^{\eta} \left( \bar{u} \cdot \nabla H - \nabla \eta \cdot \nabla h + H \right) \bar{\rho} \, dz \quad \text{(Du)}
\]

\[
\frac{g}{D} \int_{-H}^{\eta} \bar{w} \bar{\rho} \, dz = gD \nabla h - H \bar{\rho} \, dz \quad \text{(**)}, \quad \text{note that} \ \bar{\eta} \frac{\partial \eta}{\partial t} = -\nabla \bar{h} \cdot (Du).
\]

Then eq. (11) could rewrite as following:
Combining (A1) to (A5), equal to term (A) in equation (14) of B & H (2008), i.e.,

\[
\frac{g}{D}\frac{\partial n}{\partial t} \int_{-H}^{n} z \tilde{p} \, dz + \frac{g}{D} \int_{-H}^{n} z \nabla_{H} \tilde{p} \, dz + \frac{g}{D} \eta \tilde{p} \int_{-H}^{n} z \tilde{u} \, dz + \frac{g}{D} \eta \int_{-H}^{n} \tilde{p} \nabla_{H} \eta + \frac{g}{D} \frac{H \tilde{p}_{H} - \tilde{p}}{H \tilde{p}_{H} - \tilde{u} \nabla_{H} H} \\
= - \frac{g}{D^2} \nabla_{H} \cdot (D \tilde{u}) \int_{-H}^{n} z \tilde{p} \, dz + \frac{g}{D} \int_{-H}^{n} z \nabla_{H} \tilde{p} \, dz + \frac{g}{D} \eta \int_{-H}^{n} \tilde{p} \nabla_{H} \eta + \frac{g}{D} \nabla_{H} \tilde{p} \int_{-H}^{n} z \tilde{u} \, dz \\
= - \frac{g}{D^2} \left( \tilde{u} \cdot \nabla_{H} D + D \nabla_{H} \cdot \tilde{u} \right) \int_{-H}^{n} z \tilde{p} \, dz + \frac{g}{D} \int_{-H}^{n} z \nabla_{H} \tilde{p} \, dz
\]

So far, we have all the terms in equation (14) of B & H (2008).

Note that we have not used kinematic boundary conditions given by equation (16) in deriving equation (14).