Chapter 1

Phenomenology of turbulence

1.1 Coffee with milk?

Turbulence is a phenomenon present in all fluids. It is difficult to define, since it is not a material property or a state of a certain volume of fluid. It is a flow phenomenon which covers a wide range of temporal and spatial scales, it is unpredictable in a deterministic way (although statistically it obeys various laws), and most important and therefore topic of a whole lecture, it mixes.

Let us start with us creating turbulence every morning – for the purpose of increased mixing. This is at least true for those of us who drink their coffee or tea with milk or cream. The fluid consisting of milk and coffee (tea accordingly), has a molecular diffusivity, which somehow depends on the temperature, but does not change with the velocity structure in the fluid. Our goal, after pouring milk into the coffee, clearly is to create a homogeneous, well-mixed fluid composed of milk and coffee. So, why do we not wait until diffusion does its job and mixes the flow. Experience shows that this would take too long time, and the coffee would be cold before it is mixed. We know well (from the hydrodynamics lecture) that the diffusive flux is the product of the diffusivity and the gradient of the property to be mixed. Here, the gradient would be defined by differences in the milk content. Only, if it would be down to zero everywhere in the coffee mug, we would call the mixing complete. After pouring the milk in, we basically have the two liquid fractions fully separated, and a sharp gradient would only exist at the milk-coffee interface. There mixing will be strong. What do we do now, without any pre-knowledge about turbulent mixing? We take the spoon and stir the coffee-milk fluid. And within a few seconds the fluid is mixed, a process which would take hours without stirring. What happens when we stir the fluid is that we vastly increase the diffusive flux by several orders of magnitude. When denoting the diffusivity with $\nu'$ (assuming that it is constant) and the milk concentration with $m$ (in volume percent), then the local diffusive flux is the vector $-\nu' \nabla m$. Since the direction of the diffusive flux does not have an influence on the mixing (it is always down-gradient), the total diffusive flux in the coffee mug is denoted by

$$F_D = \nu' \int \int \int_V |\nabla m| dV,$$

with $V$ denoting the volume of the fluid. If our mug is cubic and has a side length of 5cm, and we have milk and coffee in equal parts, then the interface between milk and coffee has a surface of $25 \text{ cm}^2$. Let us assume that after stirring the melange with a spoon (and without mixing), we have fractionated the fluid into milk and coffee cubes of 1 mm side length, then the total surface between milk and coffee is $3 \cdot 49 \cdot 25 \text{ cm}^2 = 3675 \text{ cm}^2$, and mixing will be increased by a factor of 147. By stirring the fluid with the spoon for a couple of seconds, this fractionation will be repeated several times and within a few seconds, the coffee is ready for a drink.
Thus, we simply stir the coffee in order to sharpen the gradients and to increase the integral contact surface between high and low milk concentrations. It seems as if we would have increased the diffusivity of our milk coffee, but we didn’t. Therefore, the increased mixing due to turbulence is attributed as mediated by a so-called eddy diffusivity.

*From stirring to mixing,* this is standing expression in turbulence theory, since first you need to stir a fluid (e.g. by means of a spoon) in order to increase gradients, and then the basically constant viscosity acts on these gradients and thus induces increased mixing.

The process of mixing in the ocean works similarly, but here the stirring is mainly generated by wind, waves, cooling and evaporation at the surface, friction at the sea bed, internal friction between water masses, breaking of internal waves, and static instabilities generated inside the flow (denser water moves on top of lighter water). In the following section, we will discuss how little the ocean would mix without turbulence.

### 1.2 How efficient is molecular mixing in the ocean?

Let us assume that oceanic mixing can be explained by vertical molecular mixing, only. If we take the Navier-Stokes equations (2.19) and assume a horizontally homogeneous uni-directional flow, then the following diffusion equation will be valid:

\[ \partial_t u - \nu \partial_{zz} u = 0 \]  

with the velocity \( u = u(z,t) \) and the molecular viscosity \( \nu \). An analytical solution for (1.2) may be found by constructing the Fourier transform of \( u \):

\[ u(z,t) = \text{Re} \left( \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(ikz)U(k,t) \, dk \right) \]  

with \( i = \sqrt{-1} \) and the wave number \( k \). The derivatives of \( u \) may now be expressed as

\[ \partial_t u(z,t) = \text{Re} \left( \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(ikz)\partial_t U(k,t) \, dk \right) \]  

and

\[ \partial_{zz} u(z,t) = \text{Re} \left( \int_{-\infty}^{\infty} \frac{1}{2\pi} (-k^2) \exp(ikz)U(k,t) \, dk \right) \]  

such that

\[ \partial_t u - \nu \partial_{zz} u = \text{Re} \left( \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(ikz) \left( \partial_t U(k,t) + \nu k^2 U(k,t) \right) \, dk \right) = 0, \]  

such that

\[ \partial_t U(k,t) + \nu k^2 U(k,t) = 0 \]  

for all \( k \). Thus, we obtain

\[ U(k,t) = U(k,0) \exp(-\nu k^2 t). \]  

For \( U(k,0) = 1 \), the result is thus:

\[ U(k,t) = \exp(-\nu k^2 t). \]  

The velocity may now be calculated by means of a back transformation using (1.3):

\[ u(z,t) = \text{Re} \left( \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(ikz) \exp(-\nu k^2 t) \, dk \right) \]

\[ = 2 \int_{0}^{\infty} \frac{1}{2\pi} \cos(kz) \exp(-\nu k^2 t) \, dk, \]  

\[ u(z,t) = \text{Re} \left( \int_{-\infty}^{\infty} \frac{1}{2\pi} \exp(ikz) \exp(-\nu k^2 t) \, dk \right) \]
where the latter transformation is due to the Euler formula
\[
\exp(iz) = \cos z + i \sin z.
\] (1.11)
According to Bronstein and Semendjajew [1989], the latter integral in (1.10) results in
\[
u_{t} = \sqrt{\nu} \exp\left(-\frac{z^2}{4\nu t}\right). \tag{1.12}
\]
(1.12) describes the spatial and temporal evolution of the velocity profile when initially all velocity is concentrated in one point at the surface at \(z = 0\). It is now instructive to calculate the time needed until the velocity at a certain depth \(D\) is half the velocity at the surface. This is done by the following relation:
\[
\frac{1}{2} = \frac{u(-D,t)}{u(0,t)} = \exp\left(-\frac{D^2}{4\nu t}\right) \tag{1.13}
\]
which is equivalent to
\[
t = -\frac{D^2}{4\nu \ln(0.5)}. \tag{1.14}
\]
With the molecular viscosity of \(\nu = 1.8 \cdot 10^{-6} \text{ m}^2 \text{s}^{-1}\) and a depth of \(D = 30 \text{ m}\) (a typical surface mixed layer depth), we obtain \(t \approx 1.8 \cdot 10^8 \text{ s} \approx 2087 \text{ days} \approx 5.7 \text{ years}\). The evolution of the velocity profile is demonstrated in figure 1.1.
Since the oceanic mixed layer usually has strong annual signal (see figure 1.2) and even may change its structure on much shorter time scales (such as a couple of hours due to storms, see figure 1.3), there must be other mechanisms which mix the ocean much more efficiently than molecular mixing. It should be noted that the molecular diffusivities for temperature and salinity are even about one and three orders of magnitude smaller, respectively, such that molecular diffusion of these quantities is even much less efficient. This other process is the turbulent motions in the water about which this lecture is dealing in the following.

1.3 Observational evidence of turbulence

1.3.1 Laboratory experiments
Osborne Reynolds ([Reynolds 1883]) discovered that the flow resistance of water through tubes was below a certain critical flow velocity proportional to the flow velocity, and above this critical velocity, the resistance was proportional to the square of the flow velocity. He found that this critical velocity depended on the diameter of the tube and the viscosity of the water (which is a function of the water temperature). He further noticed by visual inspection of streaks of coloured water inside the clear water, that at this critical velocity a transition from direct (straight, laminar) motion to sinuous (turbulent) motion took place, see figure 1.4.

Reynolds [1883] found in his tube experiments that the non-dimensional number
\[
K = \frac{dv}{\nu} \tag{1.15}
\]
calculated from the product of tube diameter \(d\) with mean flow velocity \(v\), divided by the water viscosity \(\nu\), clearly determines the onset of turbulent oscillations. For \(K\) being smaller than 1900, he found laminar flow and for \(K\) being larger than 2000, the flow was turbulent.
Another interesting laboratory experiment for demonstrating the onset of turbulence in a flow is the uniform flow past a cylinder vertically erected in the flume (see figure 1.5 for a sketch of this setup). For
Figure 1.1: Evolution of a velocity profile (normalised by the surface velocity) according to the analytical solution (1.12) of the diffusion equation (1.2) with molecular viscosity $\nu$.

Figure 1.2: Observed temperature evolution for Ocean Weather Ship Papa in the Northern Pacific Ocean at 145°W, 50°N from March 1961 to March 1962 from CTD (Conductivity-Temperature-Density) measurements. This figure has been taken from Burchard [2002].
Figure 1.3: Observed temperature evolution for the Fladenground Experiment (FLEX’76) site at 58°55’N and 0°32’E in the Northern North Sea (April 6 - June 8, 1976) from CTD measurements. This figure has been taken from Burchard [2002].

Figure 1.4: Laminar (top panel) and turbulent (bottom panel) flow from left to right through a glass tube, visualised by streaks of coloured water. This figure has been taken from Reynolds [1883].
very low flow velocity, the flow is symmetrically arranged around the cylinder with streamlines smoothly aligned. At this stage, the flow is to be addressed as laminar flow. With more increasing flow velocity a wake behind the cylinder is developing. The next stage with the flow even more accelerating is that a Kármán vortex street of alternating vortices develops. At even higher flow velocities, the flow behind the cylinder becomes irregular and random, or in other word, turbulent. Nice pictures of this transition from laminar to turbulent flow can be seen in the book by van Dyke [1982].

Also for this experiment of flow around a cylinder, a dependence on the product of a length scale (here the diameter of the cylinder) and the flow velocity divided by the water viscosity can be observed. The wake behind the cylinder developed for this number larger than 10, the Kármán vortex street for this number of larger than 100. For this number being larger than 1000, the flow is clearly turbulent.

Interestingly, the transition from laminar to turbulent flow occurs at the same order of magnitude of this number. Therefore, a generalisation of (1.15) can be made: When denoting $V$ as the velocity scale and $L$ as the length scale, then we can define the non-dimensional number

$$Re = \frac{VL}{\nu},$$

(1.16)

which has been named after Osborne Reynolds the Reynolds number. The Reynolds number represents the ratio of the inertial forces to the viscous forces. Dominance if viscous forces tends to flatten out the turbulent eddies due to non-linear flow instabilities.

To calculate the Reynolds numbers for oceanic situations, is surely a much more difficult task, especially when stratification plays a role. But for a typical unstratified flow near the coast, we could take the Darss Sill as an example with a depth of 20 m as length scale, a typical flow velocity of $0.2 \text{ m s}^{-1}$ as velocity scale and the molecular viscosity of water of about $10^{-6}\text{ m}^2\text{s}^{-2}$, then we would get a Reynolds number of $4 \cdot 10^9$, which surely is a sign for fully turbulent conditions.

Often, laboratory experiments are carried out in order to be able to study sensitivities of processes under standardised conditions. The spatial and temporal scales of these experiments are adjusted such that
relevant non-dimensional parameters are the same than in the full scale of the ocean or the atmosphere. As an example, we consider an unstratified flow of $D = 10$ m depth, with a flow velocity of $u = 0.1$ m s$^{-1}$, which gives a Reynolds number of about $5 \cdot 10^5$, and a Froude number of $F_r = u/\sqrt{gD}$ (with the gravitational acceleration $g = 9.81$ m s$^{-2}$), i.e. $F_r \approx 0.01$. With the thickness of the laboratory flow of 0.1 m, and the same Froude number, we have a flow velocity of 0.01 m s$^{-1}$, and thus a Reynolds number of $Re = 5 \cdot 10^2$, i.e. a Reynolds number which does not indicate fully developed turbulence. Since under these conditions the flow velocities may be completely different than in the full scale, laboratory experiments are always difficult to interpret.

In the next section, we will have a close look into turbulent phenomena in the ocean.

1.3.2 Oceanic observations

In this section evidence of oceanic turbulence is given from observations with different kinds of instruments, showing different quantities at different locations and under different flow regimes.

As discussed for the example of the cup of coffee in section 1.1 we need turbulent mixing in order to mix more efficiently than by molecular mixing. And as demonstrated for the wind mixed upper ocean in section 1.2, molecular mixing is not sufficient to explain wind driven mixing of the upper ocean.

Therefore, we expect significant turbulent fluctuations in the mixing areas of the ocean.

Let first have a look into the velocity structure of the oceanic mixing layers$^1$. The small scale velocity structure can be measured by shear probes with an airfoil-shaped cone at the peak, sensing cross forces, when the profiler is actively rising or falling vertically, for details, see section 6.2. An example for such an observation is given in figure 1.6 which shows the small-scale vertical velocity shear in the upper mixed layer of the Baltic Sea. On the scale of the mixed layer depth, the profile looks very spiky, indicating intense small-scale turbulence. However, on the small scale of a few centimetres, we see that the profile is smooth, demonstrating the effect of the viscous forces. We see here a typical length scale of the velocity fluctuations between 3 and 4 cm. This is the scale on which the mixing actually takes place in this case.

We would expect that the smallest eddy size will decrease with increasing wind stirring.

The next example is from shallow coastal waters at the east coast of North America. Since a couple of years, the modern technique of Particle Image Velocimetry (PIV) has been applied to the ocean as well. Natural particles (e.g. zooplankton) are illuminated by a Laser sheet and then taken as double exposure photographs with short time in between. For all particles which can be traced, a velocity vector can be calculated, and from all available such vectors, vector maps as show in figure 1.7 can be processed. In contrast to the shear probe profile shown in figure 1.6, two-dimensional structures can be observed here. In this example, we see that the eddy of 5 cm diameter seen in the centre of the left panel has been advected to the right edge of the right panel, and another eddy has entered from the left edge of the left panel. This method for observing small-scale turbulence is very labour-intensive and based on heavy instrumentation (including a 10 m long garbage truck press which has to be lowered to the ground), such that micro-structure shear probes are still the most effective instrument for such observations.

Larger eddies cannot be seen by the instruments discussed above with sufficient accuracy. In recent years acoustic Doppler methods have been improved such that they can fill this gap under certain circumstances, see section 6.1. High frequency sound is emitted by so-called Acoustic Doppler Current Profilers (ADCPs), which are refracted back by particles (zooplankton) and received by the instrument again. From the Doppler shift, velocity profiles can be calculated. Figure 1.8 shows an example observed in the York River, a tributary to the Chesapeake Bay at the east coast of North America. The data have a resolution in time of 1 s and in space of 1/2 m, such that only large eddies can be resolved. We clearly see turbulent fluctuations of the along-stream velocity component with an amplitude of about 0.05 m/s. Eddy intensity

$^1$Just for later reference: a mixed layer is defined in the sense that the vertical gradients of various properties like temperature and salinity are very small. It may be mixed by a previous mixing event or may be actively mixed. In contrast to that, a mixing layer is just actively mixing.
Figure 1.6: Micro-structure shear in s\(^{-1}\) from the Northern Baltic Sea as observed by means of a micro-structure shear probe. Here, \(\tilde{u}\) is the instantaneous current speed component parallel to the direction of the instruments sensitivity. Left panel: profile over a range of 5 m; middle panel: blow-up of same profile over a range of 1 m, right panel: blow-up of same profile over a range of 0.2 m. Courtesy of Adolf Stips, Ispra (Italy).

seems to be higher in mid-water than near the bed, which is explainable with the bed friction and the limitation of vertical fluctuations near the bed.

In the example of the coffee cup, it were so-called tracers (scalars) which were mixed increasingly by turbulence. For this, fluctuations of the tracer concentrations are necessary. The most simple tracer fluctuation to measure is the temperature. For this, highly sensitive fast and small thermistors are used. Figure 1.9 shows time series of temperature measured from fixed position near the shore of Zingst (Germany). The time series in the main frame shows high-resolution temperature, the inset shows a temperature times series close to this position observed by a slow sensor, which filters out the fluctuations (but is generally more accurate in measuring the absolute temperature). The fluctuations shown in the inset represent large eddy fluctuations, since their time scale seems to be of the order of one minute.

A similar picture is seen in figure 1.10 where a vertical profile in 25 m deep water of the Mecklenburg Bight (Germany) shows coarse- fine- and micro-structure of temperature, observed from a free-falling profiler. The temperature inversion at 7 m depth was compensated by a stable salinity gradient.

In figure 1.11 a high-resolution temperature profile in winter from Lago Maggiore (Italy) is shown. Closer to the surface, the temperature is slightly lower, indicating strong mixing due to static instability (convective mixing). Various scales of eddies are visible, at about 5 m depth we see a large eddy of 5 metres vertical scale, there are a lot of eddies of some tens of centimetres scales, and there are also eddies of even smaller scales which are here not resolved by the graphics. In fact, from visual inspection of this profile, we would assume a whole spectrum of length scales.
Figure 1.7: Two velocity vector maps of the same area, sampled 1 s apart. The instantaneous mean velocity of the sample area (shown at the top of each map) is subtracted from each vector to highlight the turbulence structure. The vertical coordinates represent the actual distance from the bottom. These frames were observed by means of Particle Image Velocimetry (PIV) near the LEO-15 site off the coast of New Jersey in 15 to 21 m depth of water. Figure by Alex Nimmo Smith (Plymouth, England), see also Nimmo Smith et al. [2002].
Figure 1.8: Time series (10 minutes) of instantaneous current velocities observed in the York River estuary (Virginia, U.S.A.) along one beam of a high-resolution Acoustic Doppler Current Profiler (ADCP) at five heights above the bed. Data have been logged every second, the distance between two beams is 0.5 m, bin 1 is near the bed, bin 5 is at mid depth. Note that not all the fluctuations are due to turbulence: the uncertainty in each measurement is about 0.014 m s$^{-1}$. Data by Eirwen Williams (Bangor, Wales), see also Williams and Simpson [2004].
Figure 1.9: Time series of mean water temperature (above) and high-resolution water temperature (below) on August 11, 1975 at a pile station near the shore of Zingst (Germany) at a water depth of about 3 m. This figure has been taken from Baudler [1978].

Figure 1.10: Coarse-, fine- and micro-structure of the temperature stratification at a station in the Mecklenburg Bight as observed on July 16, 1981. The figure has been taken from Prandke [1983].
Figure 1.11: High-resolution temperature profile taken in lake Lago Maggiore in December 1995 during strong convective mixing with a free-rising profiler. Data by Adolf Stips (Ispra, Italy), see also Stips et al. [2002].
1.4 Definition of turbulence

When dealing with turbulence, one major problem is that an exact definition does not exist. What turbulence means, is thus better identified by listing some properties by means of which turbulent flow are defined (see Lesieur [1997]):

1. **Randomness:** A turbulent flow must be unpredictable, in the sense that a small uncertainty as to its knowledge at a given initial time will amplify as to render impossible a precise deterministic prediction of its evolution. This seems to be somewhat in contradiction to the fact that turbulent flows are to a large degree sufficiently described by the Navier-Stokes equations which are partial differential equations with necessary initial and boundary conditions (see section 2.2). These Navier-Stokes equations are clearly deterministic. However, due to their non-linearity, the Navier-Stokes equations contain instabilities which strongly amplify small disturbances with time. Giving the uncertainty of the initial conditions and the noise of the Brownian motion of water molecules (which is not considered in the Navier-Stokes equations due to the continuum hypothesis) an instantaneous turbulent fluctuations is indeed unpredictable. However, in a statistical sense, turbulence is predictable. The phenomenon of unpredictable dynamics resulting from a deterministic problem is called deterministic chaos see section 1.5. Weather is a typical example for this.

2. **Diffusivity:** It has to mix transported quantities much more rapidly than only molecular diffusion would do. This property of turbulence has probably the most important consequences: turbulence is characterised by enhanced mixing. This mixing may be some orders of magnitude higher than the mixing due to molecular diffusion. Whilst the molecular diffusion coefficient for heat is about $10^{-7} \text{m}^2\text{s}^{-1}$, typical values of turbulence-induced mixing coefficients (the so-called eddy diffusivity) on the ocean surface mixed layer are of the order of $10^{-2} \text{m}^2\text{s}^{-1}$. At this point, it might be helpful to discuss qualitatively why turbulence does enhance mixing. For this, figure 1.12 demonstrates how turbulence does increase the interface between two areas of different concentration.

3. **Vorticity:** Turbulent flows are characterised by high values of vorticity, i.e. rotating eddies. These eddies do involve a wide range of spatial wave lengths, ranging from the size of the boundary layers (e.g. surface or bottom) to the smallest size were the eddies are dissipated due to viscosity. This is already visible in the figures 1.6 – 1.10. Various spatial scales of turbulence are involved. The way how the energy of the fluctuations is distributed among the wave lengths is discussed in the chapter 4 on turbulence spectra.

4. **Dissipation:** Turbulence is dissipative, i.e. that kinetic energy is dissipated into heat at the small eddy sizes. In order to maintain a flow to be turbulent, energy has thus to be put in into the flow, which usually happens at the larger scales.

5. **Nonlinearity:** Turbulent flow is essentially nonlinear which causes the flow to be unstable when the non-linearities grow. Thus, small perturbances grow spontaneously, see section 1.5.

1.5 Turbulence as chaotic nonlinear system

A famous example for deterministic chaos is the Lorenz [1963] model for thermal convection. For this, the following idealised experimental Bénard convection setup is considered (see Franz [1989]):

1. A horizontally infinite fluid layer of depth $H$ is heated from below;

2. The temperature profile in the unstably stratified ground state is assumed to be linear and the density to depend linearly on temperature, i.e. the temperature gradient $\Delta T$ in the basic state is assumed to be a constant;
Figure 1.12: Qualitative sketch for explaining the amplification of mixing due to turbulence. Let the different colours represent the concentration of one substance, e.g. red means high and blue low concentration. Time proceeds from left to right and constant diffusive mixing is active. The upper three panels represent small turbulence with only one eddy revolution, and lower panel higher turbulence with four eddy revolutions. It can be clearly seen that diffusive mixing acts on a much larger interface for the high turbulence such that mixing is more effective. One other effect of turbulence is that gradients are sharpened such that mixing which is proportional to the gradients is enhanced as well.

3. The motion is assumed to take place in the $x$-$z$ plane, only.

For this situation, the equations of continuity, momentum and heat transfer and the related boundary conditions (zero stress for momentum) are satisfied by the following equations, which result from the Navier-Stokes equations (2.19) by neglecting all gradients with respect to $x_2 = y$ and Earth rotation:

$$\partial_x u + \partial_z w = 0,$$

$$\partial_t u + u \partial_x u + w \partial_z u - \nu (\partial_{xx} + \partial_{zz}) u = -\frac{1}{\rho_0} \partial_z p,$$

$$\partial_t w + u \partial_x w + w \partial_z w - \nu (\partial_{xx} + \partial_{zz}) w = -\frac{1}{\rho_0} \partial_x p - \frac{g}{\rho_0} \rho,$$

$$\partial_t \theta + u \partial_x \theta + w \partial_z \theta - \kappa (\partial_{xx} + \partial_{zz}) \theta = 0,$$

with the velocity vector $(u, w)$, the potential temperature $\theta$, the pressure $p$, the potential density $\rho$, the reference density $\rho_0$, the gravitational acceleration $g$, the kinematic viscosity $\nu$, and the thermal diffusivity $\kappa$.

Let $\psi$ be the streamfunction defined by

$$u = -\partial_z \psi, \quad w = \partial_x \psi.$$

The streamfunction exists here, since the flow is incompressible:

$$\partial_x u + \partial_z w = -\partial_{xx} \psi + \partial_{zz} \psi = 0.$$
Let further the deviation from a linear temperature profile \( \tilde{\theta} \) be defined as
\[
\tilde{\theta} = \theta - \left( \theta_0 - z \frac{\Delta T}{H} \right),
\] (1.23)
with the constant depth \( H \), the mean bottom to top temperature difference and the constant reference temperature \( \theta_0 \) and the linear equation of state,
\[
\rho = \rho_0 \left( 1 + \alpha (\theta_0 - \theta) \right),
\] (1.24)
with the constant thermal expansion coefficient \( \alpha \), the system of equations can be transformed to
\[
\partial_t (\nabla^2 \psi) - \partial_z \psi \partial_x (\nabla^2 \psi) + \partial_x \psi \partial_z (\nabla^2 \psi) - \nu (\nabla^4 \psi) = g \alpha \partial_x \tilde{\theta},
\] (1.25)
and
\[
\partial_t \tilde{\theta} - \partial_z \psi \partial_x \tilde{\theta} + \partial_x \psi \partial_z \tilde{\theta} - \kappa (\partial_{xx} + \partial_{zz}) \tilde{\theta} = \frac{\Delta T}{H} \partial_x \psi,
\] (1.26)
This is obtained by first taking \( -\partial_z \) of eq. (1.18) and \( \partial_x \) of eq. (1.19) and then adding the two resulting equations. The pressure gradients on the right hand sides will then drop out. For the density \( \rho \) the equation of state (1.24) is. This will result in eq. (1.25) which is a transport equation for \( \nabla^2 \psi = (\partial_{xx} + \partial_{zz}) \psi \). Eq. (1.26) will be obtained by inserting the \( \tilde{\theta} \) equation into (1.20).
A solution of (1.25) and (1.26) may be constructed as
\[
\psi = X \kappa \sqrt{2} (1 + a^2) \sin \left( \pi a H^{-1} x \right) \sin \left( \pi H^{-1} z \right)
\] (1.27)
and
\[
\tilde{\theta} = R_a \frac{\Delta T}{\pi R_a} \left\{ Y \sqrt{2} \cos \left( \pi a H^{-1} x \right) \sin \left( \pi H^{-1} z \right) - Z \sin \left( 2 \pi H^{-1} z \right) \right\},
\] (1.28)
with the aspect ratio \( a = H/B \) (with \( B \) being the length and \( H \) being the height of the convective rolls, see figure 1.13),
\[
R_a = g \alpha H^3 \Delta T \nu^{-1} \kappa^{-1}
\] (1.29)
being the Rayleigh number, and \( R_c = \pi^2 a^{-2} (1 + a^2)^3 \) being the critical Rayleigh number, and \( X, Y \) and \( Z \) being scalar functions depending on time only. It can be shown that insertion of (1.27) and (1.28) into the linearised equation (1.25) and into the full equation (1.26) approximately results in the following scalar system of equations (for details, see exercise 1 at the end of this chapter 1):
\[
\dot{X} = P_r (Y - X),
\]
\[
\dot{Y} = -X Z + \tau X - Y,
\] (1.30)
\[
\dot{Z} = XY - b Z,
\]
with the Prandtl number of the flow, \( P_r = \kappa^{-1} \nu \) (ratio of viscosity to diffusivity), the ratio of the Rayleigh number to the critical Rayleigh number, \( r = R_c^{-1} R_a \), and \( b = 4 (1 + a^2)^{-1} \). In (1.30), the dot denotes derivatives with respect to the dimensionless time \( \tau = \pi^2 H^{-2} (1 + a^2) \). With this, it is shown that the system of equations (1.30) is physically motivated by approximating the Benard convection problem.
It can be easily seen that the system of equations (1.30) has one trivial solution with \( X = Y = Z = 0 \) corresponding to \( u = v = \tilde{\theta} = 0 \) and two non-trivial steady-state solutions \( Y = X = \pm \sqrt{b (r - 1)} \), \( Z = r - 1 \), demanding \( r > 1 \) and corresponding to steady rolls.
The numerical solution of (1.30) is shown in figure 1.14 for $b = 8/3$, $Pr = 10$ and $r = 28$. These are the same values as chosen by Lorenz [1963]. It can be seen that the solution is oscillating around the two (unstable) steady states in an aperiodic manner.

Numerical calculations of this ordinary system of equations will be task of an assignment. It will come out that smallest perturbations in the initial values will result in a time series for $X$, $Y$ and $Z$ which is completely uncorrelated to the unperturbed time series from a certain point in time onwards. An analytical solution for the Lorenz equations cannot be found anyway, and all numerical solutions will contain inaccuracies with the result that numerical schemes will not converge towards a hypothetical analytical solution.
Figure 1.14: Numerical solution of the Lorenz [1963] equations (1.30). The fix points are indicated by dots in the phase plots and by dashed lines in the times series plots.