Chapter 4

Turbulence spectra

4.1 Correlations and spectra

The autocorrelation of a single variable $u(t)$ at two times t_1 and t_2 is defined as

$$R(t_1, t_2) = \langle u(t_1)u(t_2)\rangle. \tag{4.1}$$

In the case of a statistically stationary process, the autocorrelation does only depend on the difference $\tau = t_2 - t_1$ but not on the actual instants of time, t_1 and t_2. Instead of (4.1), we could for stationary processes thus also define the autocorrelation as

$$R(\tau) = \langle u(t)u(t+\tau)\rangle. \tag{4.2}$$

With this, we define the normalised autocorrelation function as

$$r(\tau) = \frac{\langle u(t)u(t+\tau)\rangle}{\langle u^2\rangle}. \tag{4.3}$$

For any function $u(t)$, it can be shown that

$$\langle u(t_1)u(t_2)\rangle \leq \langle u^2(t_1)\rangle^{1/2}\langle u^2(t_2)\rangle^{1/2}, \tag{4.4}$$

(which is the Schwartz inequality) and thus for a stationary process we obtain thus $r \leq 1$. Figure 4.1 demonstrates how the autocorrelation of a statistically stationary time series can be calculated. A typical autocorrelation plot is shown in figure 4.2. For the time lag $\tau = 0$, the autocorrelation is trivially unity, and it decays with increasing time lag. For the limit of very large time lags the autocorrelation function converges to zero, since turbulence is a random phenomenon. This allows for the definition of an integral time scale $\mathcal{T}$ of turbulence:

$$\mathcal{T} = \int_0^\infty r(\tau)\, d\tau, \tag{4.5}$$

see figure 4.2 for a graphical demonstration. $\mathcal{T}$ represents the time over which the process $u(t)$ is highly correlated to itself.
Let now $\tilde{u}_i(t)$ be the velocity fluctuation vector in a point. Then, more general correlations are given by

$$R_{ij}(\tau) = \langle \tilde{u}_i(t)\tilde{u}_j(t+\tau)\rangle \tag{4.6}$$

which is the two-point crosscorrelation between u_i and u_j. For $\tau = 0$ we obtain the one-point cross-correlation which is symmetric and identical to the Reynolds stress $R_{ij}(0) = \langle \tilde{u}_i\tilde{u}_j\rangle$ defined in section 3.2.

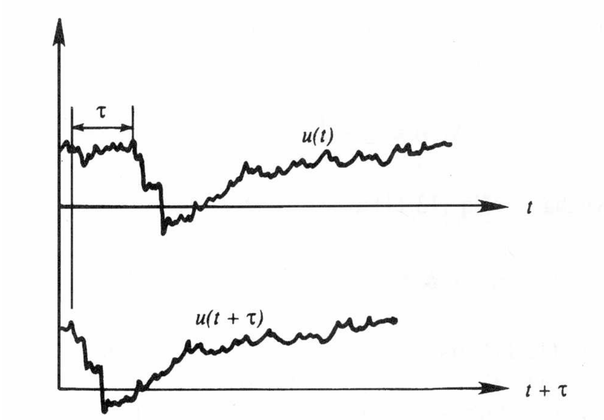

Figure 4.1: Method of calculating the autocorrelation $R(\tau) = \langle u(t)u(t+\tau)\rangle$. This figure has been taken from *Kundu and Cohen* [2002].

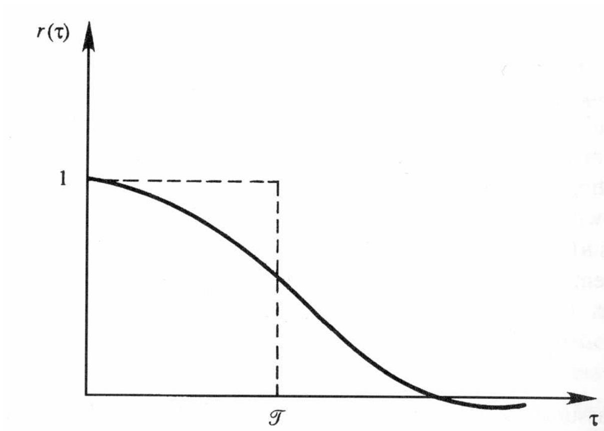

Figure 4.2: Autocorrelation function $r(\tau)$ and the integral time scale $\mathcal{T}$. This figure has been taken from *Kundu and Cohen* [2002].

Let $E_{ij}(\omega)$ denote the Fourier transform of the correlation function $R_{ij}(\tau)$:

$$E_{ij}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega\tau} R_{ij}(\tau)\, d\tau \tag{4.7}$$

with the frequency ω. It can be shown that (4.7) is only properly defined if it an be reversed:

$$R_{ij}(\tau) = \int_{-\infty}^{\infty} e^{i\omega\tau} E_{ij}(\omega)\, d\omega. \tag{4.8}$$

In this case, the equations (4.7) and (4.8) define a *Fourier transform pair.*
For $\tau = 0$ and $i = j$, we immediately see from (4.8) that

$$\langle \tilde{u}_i^2 \rangle = \int_{-\infty}^{\infty} E_{ii}(\omega)\, d\omega, \tag{4.9}$$

and

$$k = \int_{0}^{\infty} E_{ii}(\omega)\, d\omega, \tag{4.10}$$

with the turbulent kinetic energy from (3.35), i.e. that $E_{ii}(\omega)\, d\omega$ is the turbulent kinetic energy in a frequency band $d\omega$ centered at ω. $E_{ii}(\omega)$ is thus the (three-dimensional) energy spectrum, showing how kinetic energy is distributed as a function of frequency. It is further interesting to calculate $E_{ii}(0)$:

$$E_{ii}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{ii}(\tau)\, d\tau = \frac{\langle \tilde{u}_j^2 \rangle}{\pi} \int_{0}^{\infty} r_{ii}(\tau)\, d\tau = \frac{\langle \tilde{u}_j^2 \rangle}{\pi} \mathcal{T} \tag{4.11}$$

which shows that the spectrum at zero frequency is proportional to the integral time scale.
So far, we have only considered correlations in time. However, correlations in space can be treated in an analogue way. Let $\vec{x}_0$ and $\vec{x}_0 + \vec{x}$ be two points in the three-dimensional Cartesian space. Then the spatial cross correlation of a function $\tilde{u}_i(\vec{x}, t)$ can be formulated as

$$R_{ij}(\vec{x}) = \langle \tilde{u}_i(\vec{x}_0, t) \tilde{u}_j(\vec{x}_0 + \vec{x}, t) \rangle. \tag{4.12}$$

The energy spectrum tensor $E_{ij}(\vec{K})$ with the wave number vector $\vec{K}$ can here be formulated in the same way as in eq. (4.7):

$$E_{ij}(\vec{K}) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{-i\vec{K}\cdot\vec{x}} R_{ij}(\vec{x})\, d\vec{x} \tag{4.13}$$

where

$$R_{ij}(\vec{x}) = \iiint_{-\infty}^{\infty} e^{i\vec{K}\cdot\vec{x}} E_{ij}(\vec{K})\, d\vec{K}. \tag{4.14}$$

For $i = j$ and $\vec{x} = \vec{0}$ we obtain

$$R_{ii}(\vec{0}) = \langle \tilde{u}_i^2 \rangle = \iiint_{-\infty}^{\infty} E_{ii}(\vec{K})\, d\vec{K}. \tag{4.15}$$

For simplicity, the information about the direction of the Fourier modes is often removed by considering the length of the wave number vector, $K = \left(K_i^2\right)^{1/2}$ instead of the vector itself. This is obtained by integrating over the surfaces of spheres $\mathcal{S}(K)$ with radius K, such that we can write instead of (4.15):

$$k = \frac{1}{2}\langle \tilde{u}_i^2 \rangle = \iiint_{-\infty}^{\infty} \frac{1}{2} E_{ii}(\vec{K})\, d\vec{K} = \int_0^{\infty} \oint \frac{1}{2} E_{ii}(\vec{K})\, \mathrm{d}\mathcal{S}(K)\, dK. \tag{4.16}$$

With defining

$$E(K) = \oint \frac{1}{2} E_{ii}(\vec{K}) \mathrm{d}\mathcal{S}(K), \tag{4.17}$$

we obtain

$$k = \frac{1}{2}\langle \tilde{u}_i^2 \rangle = \int_0^{\infty} E(K)\, dK. \tag{4.18}$$

$E(K)$ is thus the scalar energy distribution function per mass unit for a certain wave number value. When investigating turbulent spectra, often the assumption of isotropy is made. This implies that all gradients of averaged quantities vanish and that the Reynolds stress may be expressed as

$$\langle \tilde{u}_i \tilde{u}_j \rangle = \frac{2}{3} k \delta_{ij}. \tag{4.19}$$

As a consequence of this, the turbulent kinetic energy equation (3.36) reduces to

$$\partial_t k = -\varepsilon. \tag{4.20}$$

For homogeneous turbulence, a dynamic equation for the three-dimensional spectrum can be derived by means of Fourier transforming the dynamic equation for the two-point cross correlation R_{ij} (see eq. (4.12)), which can be seen as an extention of the Reynolds stress equation (3.29), see *Hinze* [1975] for details:

$$\partial_t E(K) + \partial_K \mathcal{T}(K) = \mathcal{P}(K) - 2\nu K^2 E(K) \tag{4.21}$$

where $\mathcal{T}(K)$ is the spectral energy transfer rate, i.e. the transport of energy from the lower wave number to the higher wave numbers (the so-called spectral energy cascade). $\mathcal{P}(K)$ is the production of turbulence at wave number K, which vanishes for isotropic turbulence dominating the smaller scales, such that turbulence production is dominant on the larger scales. The last term in (4.21) is the spectral formulation of the turbulent dissipation rate.

Thus, for isotropic turbulence, we obtain $\mathcal{P}(K) = 0$ and since $\mathcal{T}(0) = \mathcal{T}(\infty) = 0$, integration of (4.21) over the whole spectrum leads to

$$\int_0^{\infty} \partial_t E(K)\, dK = \partial_t \int_0^{\infty} E(K)\, dK = \partial_t k = \varepsilon. \tag{4.22}$$

With (4.21), we obtain a second formulation for the dissipation rate for the turbulent kinetic energy:

$$\varepsilon = 2\nu \int_0^{\infty} K^2 E(K)\, dK. \tag{4.23}$$

It was shown in section 3.3 that the pseudo-dissipation is equal to the dissipation rate for homogeneous turbulence, see equation (3.38): $\varepsilon = \nu \langle (\partial_j \tilde{u}_i)^2 \rangle$. Since on the spatial scales at which the dissipation is most important local isotropy can be assumed (operators are invariant to rotation and reflection), only two types of gradients contribute to the dissipation rate, the longitudinal gradients, e.g. $\partial_x \tilde{u}$ and the transversal gradients, e.g. $\partial_z \tilde{u}$. Since the autocorrelations of these two types are statistically proportional, i.e.

$$\left\langle (\partial_x \tilde{u})^2 \right\rangle \propto \left\langle (\partial_z \tilde{u})^2 \right\rangle, \tag{4.24}$$

the dissipation rate is also proportional to these correlations. In detail one can show for isotropic turbulence:

$$\frac{1}{15}\varepsilon = \nu\left\langle(\partial_x\tilde{u})^2\right\rangle, \quad \frac{2}{15}\varepsilon = \nu\left\langle(\partial_z\tilde{u})^2\right\rangle, \tag{4.25}$$

such that with three correlators of longitudinal type, $\nu\left\langle(\partial_x\tilde{u})^2\right\rangle$, and six correlators of transverse type, $\nu\left\langle(\partial_z\tilde{u})^2\right\rangle$, this adds up to ε (see exercise 5.28 in *Pope* [2000]).

In order to show the relation between power and dissipation spectra, we can thus consider the one-dimensional case with homogeneous and isotropic turbulence. Let us define for this case the two-point correlation

$$\langle\tilde{u}(x)\tilde{u}(x+s)\rangle = R(s) = \int_{-\infty}^{\infty} e^{-iKs}E(k)\mathrm{d}K, \tag{4.26}$$

with the energy spectrum $E(K)$.

In the same way, we can define the two point correlation for the gradients of $\tilde{u}$:

$$\langle\partial_x\tilde{u}(x)\partial_x\tilde{u}(x+s)\rangle = B(s) = \int_{-\infty}^{\infty} e^{-iKs}D(K)\mathrm{d}K. \tag{4.27}$$

For this isotropic case, we obtain with (4.24):

$$\varepsilon \propto \lim_{s\to 0} B(s), \tag{4.28}$$

such that we can interpret $D(K)$ as the dissipation spectrum. We are now interested, how $E(K)$ and $D(K)$ are related to each other. For this, we use the simple equality

$$\partial_x\tilde{u}(x+s) = \partial_s\tilde{u}(x+s). \tag{4.29}$$

Due to homogeneity, we have

$$\partial_x\langle\tilde{u}(x)\partial_x\tilde{u}(x+s)\rangle = 0 \quad \Rightarrow \quad \langle\partial_x\tilde{u}(x)\partial_x\tilde{u}(x+s)\rangle = -\langle\tilde{u}(x)\partial_{xx}\tilde{u}(x+s)\rangle. \tag{4.30}$$

With this, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-iKs}D(K)\mathrm{d}K &= B(s) \\
&= \langle\partial_x\tilde{u}(x)\partial_x\tilde{u}(x+s)\rangle \\
&= -\langle\tilde{u}(x)\partial_{xx}\tilde{u}(x+s)\rangle \\
&= -\langle\tilde{u}(x)\partial_{ss}\tilde{u}(x+s)\rangle \\
&= -\partial_{ss}\langle\tilde{u}(x)\tilde{u}(x+s)\rangle \\
&= -\partial_{ss}R(s) \\
&= -\partial_{ss}\int_{-\infty}^{\infty} e^{-iKs}E(K)\mathrm{d}K \\
&= \int_{-\infty}^{\infty} e^{-iKs}K^2E(K)\mathrm{d}K,
\end{aligned} \tag{4.31}$$

such that

$$D(K) = K^2 E(K). \tag{4.32}$$

The same could have been carried out with $\langle \partial_z \tilde{u}(z) \partial_z \tilde{u}(z+s) \rangle$. Thus, we dissipation spectrum is obtained from the power spectrum by simply multiplying with the square of the wave number.
Now, we have a complete picture of the energy cascade process as it is depicted in figure 4.3.
In order to calculate $R_{ij}(\vec{x})$, we would need simultaneous observations of $u(\vec{x})$ in a volume of space. First of all, this is currently close to impossible in the ocean. It is thus often assumed that the turbulence in a volume of space can be represented by turbulence along a one-dimensional axis x. Also after this simplification, it is impossible to obtain simultaneous observations with instruments moving vertically or horizontally through the water (see section 6.1 for observational techniques which do produce simultaneous measurements in one or two dimensions). With such instruments, observations of $u(x)$ are approximated by assuming that the turbulence changes only little during the course of the instrument. Then, the time series $u(t)$ may be converted into a simultaneous profile $u(x)$ by setting $x = t \cdot U_0$ with U_0 denoting the profiling speed of the instrument. This assumption of *frozen* turbulence is called *Taylor's hypothesis*. Examples of such profiling systems moving through the water are discussed in section 6.2.

4.2 Three-dimensional kinetic energy spectra

Energy spectra for turbulence have been theoretically derived by *Kolmogorov* [1941]. He assumed that at sufficiently high Reynolds numbers the flow is locally homogeneous and isotropic and to be statistically in equilibrium in this range of high wave numbers. Thus, in this range of the spectrum, turbulence should uniquely be determined by dissipation rate ε and viscosity ν. By means of dimensional analysis, a characteristic length scale λ for the viscous eddies can then be defined. The involved quantities have the following units[1]:

$$(\lambda) = L \tag{4.33}$$

$$(\nu) = L^2 T^{-1} \tag{4.34}$$

$$(\varepsilon) = L^2 T^{-3}, \tag{4.35}$$

such that we obtain

$$\lambda = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}, \tag{4.36}$$

which is called the Kolmogorov microscale at which turbulence dissipates into heat due to the effect of viscosity.
The picture of turbulence is that energy is extracted from the meanflow at the low wave number range with wavenumbers of $K \approx l^{-1}$ where l is the integral length scale and K the length of the wave number vector $\vec{K}$. The turbulent kinetic energy is then cascading down to higher wave numbers by means of the process of vortex stretching due to the non-linear terms in the Navier-Stokes equations. When reaching the large wave numbers where viscosity dominates, this energy is then dissipated into heat. The range of length scales much smaller than the integral length scale is called the *equilibrium range* since there the spectrum is nearly isotropic and generally in equilibrium. This means that the energy in the equilibrium range is not depending on how much energy is present at the larger scales, but does only depend on the parameters that determine the nature of the small-scale flow, so that we can write

$$E = E(K, \varepsilon, \nu) \qquad K \gg l^{-1}. \tag{4.37}$$

[1] L represents a length in metres and T a time in seconds.

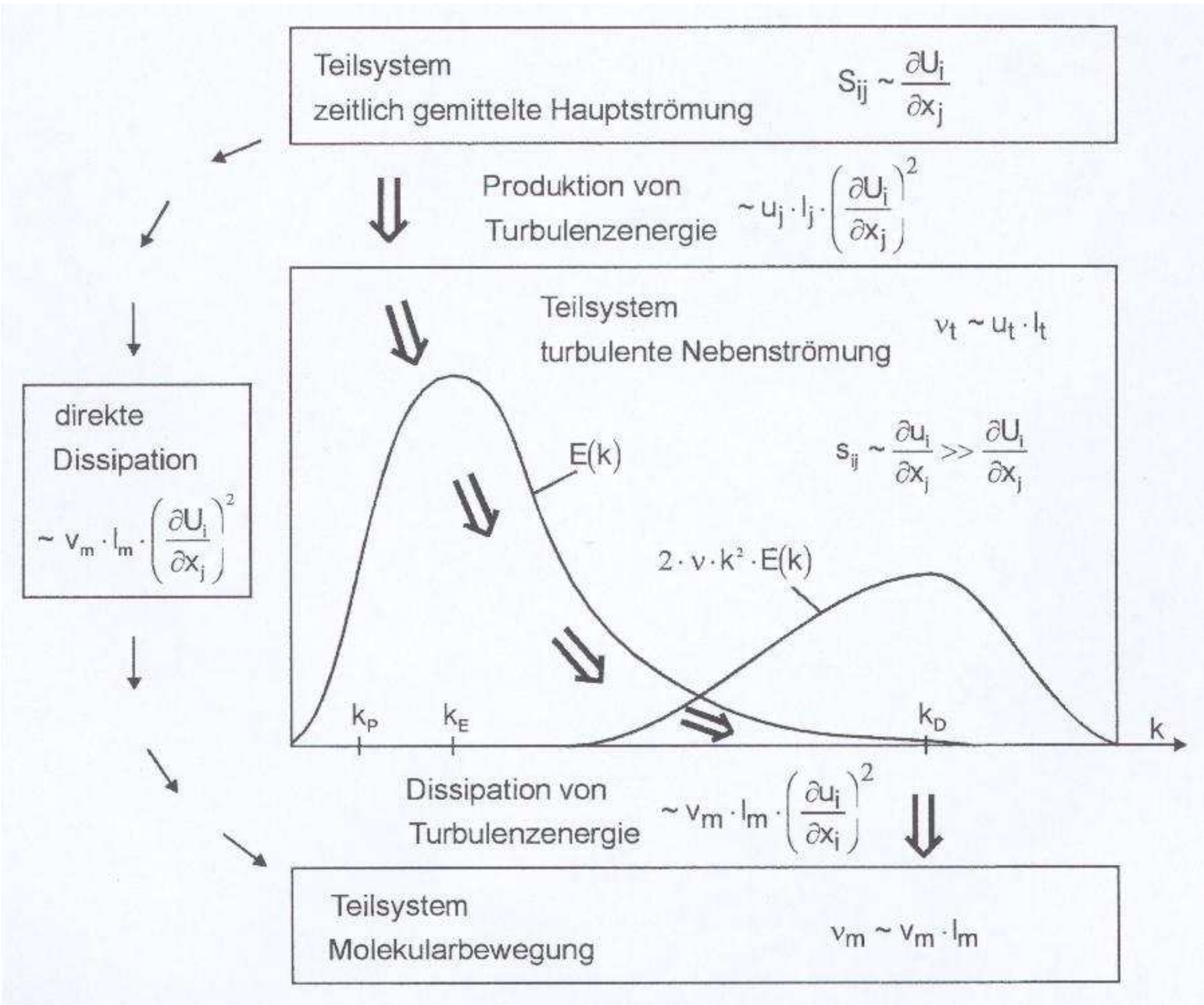

Figure 4.3: Sketch of the energy cascade process. This figure has been taken from the turbulence lecture from *Schatzmann* [1999].

The short wave number subrange of the equilibrium range which is not affected by viscosity is called the inertial subrange and can thus be described by

$$E = E(K, \varepsilon) \qquad \lambda^{-1} \gg K \gg l^{-1}. \tag{4.38}$$

Kolmogorov [1941] used dimensional analysis for theoretically deriving the shape of the spectrum in the inertial subrange.
The involved quantities have the following units:

$$(E) = L^3 T^{-2} \tag{4.39}$$
$$(\varepsilon) = L^2 T^{-3} \tag{4.40}$$
$$(K) = L^{-1}, \tag{4.41}$$

the shape of the spectrum must be

$$E(K) = K_0 \varepsilon^{2/3} K^{-5/3} \qquad \lambda^{-1} \gg K \gg l^{-1} \tag{4.42}$$

with the Kolmogorov number K_0 ranging between 1.4 and 1.8, and depending slightly on the Reynolds number R_e. Equation (4.42) is called *Kolmogorov's* $K^{-5/3}$ *law*.
For the viscous subrange dimensional analysis does not provide a solution. The general form for the whole equilibrium range may be formulated as

$$E(K) = K_0 \varepsilon^{2/3} K^{-5/3} f(K\lambda) \qquad K \gg l^{-1} \tag{4.43}$$

with the non-dimensional function f. Several formulations have been suggested for f, such as

$$f(K\lambda) = \left(1 - 0.5 K_0 (K\lambda)^{4/3}\right)^2 \tag{4.44}$$

(see *Kovasznay* [1948]),

$$f(K\lambda) = \left(1 + (1.5K_0)^2 (K\lambda)^4\right)^{-4/3} \tag{4.45}$$

(see *Heisenberg* [1948]) or

$$f(K\lambda) = \exp\left(-1.5 K_0 (K\lambda)^{4/3}\right) \tag{4.46}$$

(see *Corrsin* [1964] and *Pao* [1965]). These theoretical spectra are shown in figure 4.4. The according dissipation spectra are shown in figure 4.5.
It should be recalled that the spectra derived here are three-dimensional spectra which are generally difficult to observe in the ocean. Usually, one-dimensional profiles with stationary remote-sensing instruments or actively profiling instruments are observed, see chapter 6. Thus, one-dimensional spectra have to be considered. This is however only feasable with reasonable effort for the idealised case of isotropy. In isotropic turbulence, no preferred direction exists, such that the off-diagonal terms of the Reynolds stress tensor vanish, see figure 4.6.

4.3 One-dimensional kinetic energy spectra

The argumentation in this section follows mostly the excellent text book by *Pope* [2000].
In practise, oceanic or atmospheric observations of turbulent quantities are mostly along one dimension only. It is thus important to understand the properties of the one-dimensional spectra in relation to those of fully three-dimensional spectra. This is of course only possible when the turbulence is homogeneous and isotropic on the scales of interest, which – in turbulence – are generally the small-scales. On the

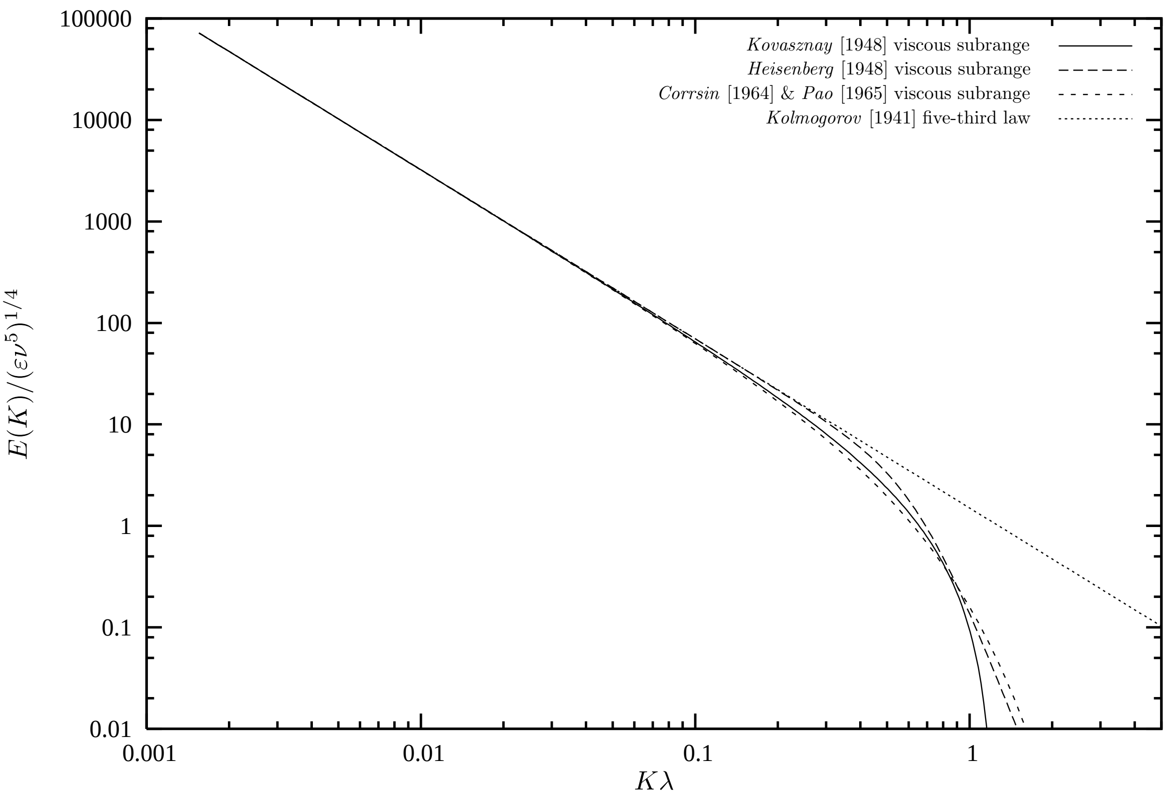

Figure 4.4: Theoretical three-dimensional turbulence energy spectra for the equilibrium range.

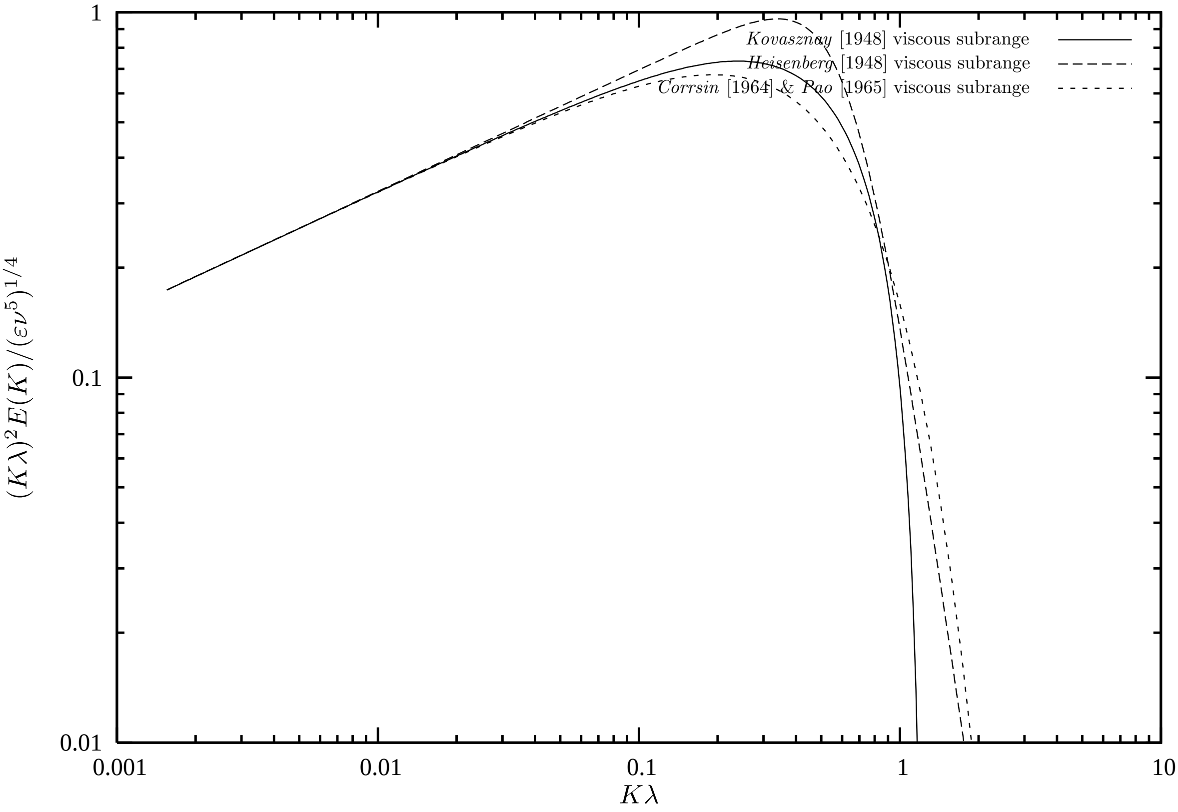

Figure 4.5: Theoretical three-dimensional turbulence dissipation spectra for the equilibrium range.

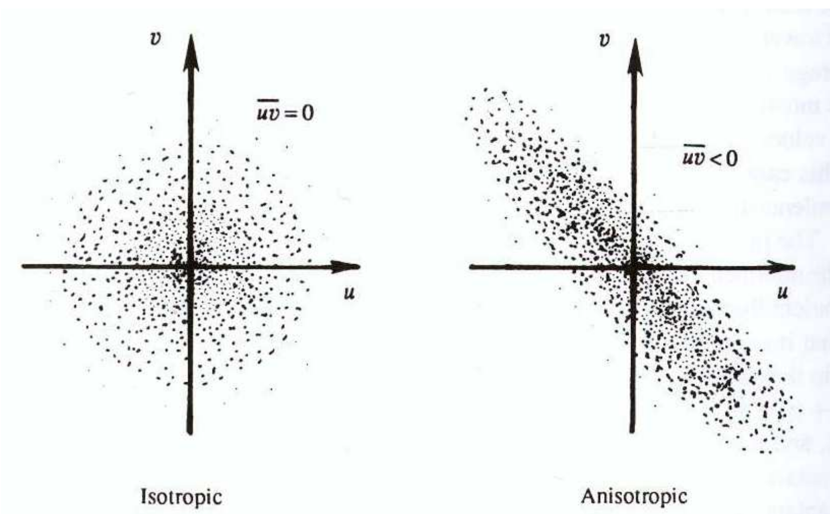

Figure 4.6: Isotropic and anisotropic turbulent fields. Each dot represents a $\tilde{u}\tilde{v}$ pair at a certain time. This figure has been taken from *Kundu and Cohen* [2002].

dissipative and the high wavenumber end of the inertial scales, local homogeneity and local isotropy can be assumed for most geophysically relevant cases.
For simplicity the axis along which a one-dimensional profile is considered is the x_1 axis. It is then distinguished between fluctuations of the velocity component paralell to this axis (longitudinal spectra) and those orthogonal to this axis (transverse spectra).
A one-dimensional spectrum is defined as:

$$E_{ij}(K_1) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_{ij}(\vec{e}_1 x_1) e^{-iK_1 x_1} \mathrm{d}x_1. \tag{4.47}$$

With this, the longitudinal spectrum is of the following form:

$$E_{11}(K_1) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_{11}(\vec{e}_1 x_1) e^{-iK_1 x_1} \mathrm{d}x_1 = \frac{2}{\pi} \int_{0}^{\infty} R_{11}(\vec{e}_1 x_1) \cos(K_1 x_1) \mathrm{d}x_1, \tag{4.48}$$

where the latter identity is due to the fact that R_{11} is real and symmetric.
For the transverse spectrum we define:

$$E_{22}(K_1) = \frac{1}{\pi} \int_{-\infty}^{\infty} R_{22}(\vec{e}_1 x_1) e^{-iK_1 x_1} \mathrm{d}x_1 = \frac{2}{\pi} \int_{0}^{\infty} R_{22}(\vec{e}_1 x_1) \cos(K_1 x_1) \mathrm{d}x_1. \tag{4.49}$$

Backtransformation results in

$$R_{11}(\vec{e}_1 x_1) = \int_{0}^{\infty} E_{11}(K_1) \cos(K_1 x_1) \mathrm{d}K_1, \tag{4.50}$$

and an equivalent result for the transverse correlation, $R_{22}(\vec{e}_1 x_1)$. Note that

$$R_{11}(\vec{0}) = \langle \tilde{u}_1^2 \rangle = \int_{0}^{\infty} E_{11}(K_1) \mathrm{d}K_1, \tag{4.51}$$

and

$$R_{22}(\vec{0}) = \langle \tilde{u}_2^2 \rangle = \int_{0}^{\infty} E_{22}(K_1) \mathrm{d}K_1. \tag{4.52}$$

In isotropic turbulence, the velocity correlation $R_{ij}(\vec{x})$ can be expressed by the two non-dimensional second order tensors δ_{ij} and $x_i x_j / x^2$ and two scalar multiples, since the only directional information is contained in the correlation distance $\vec{x}$:

$$R_{ij}(\vec{x}) = \langle \tilde{u}^2 \rangle \left(g(x) \delta_{ij} + (f(x) - g(x)) \frac{x_i x_j}{x} \right), \tag{4.53}$$

with $x^2 = x_k^2$ and $\langle \tilde{u}_i \tilde{u}_j \rangle = \langle \tilde{u}^2 \rangle \delta_{ij}$, such that $\langle \tilde{u}_i^2 \rangle = 3 \langle \tilde{u}^2 \rangle$. For isotropic turbulence, we have $\langle \tilde{u}^2 \rangle = \langle \tilde{u}_1^2 \rangle = \langle \tilde{u}_2^2 \rangle = \langle \tilde{u}_3^2 \rangle$. With setting $\vec{x} = \vec{e}_1 x_1$, i.e. $x_2 = x_3 = 0$ and $x_1 = x$, this can be transformed into

$$\begin{aligned} &\frac{R_{11}(\vec{e}_1 x_1)}{\langle \tilde{u}^2 \rangle} = f(x) = \frac{R_{11}(\vec{e}_1 x_1)}{\langle \tilde{u}_1^2 \rangle}, \\ &\frac{R_{22}(\vec{e}_1 x_1)}{\langle \tilde{u}^2 \rangle} = g(x) = \frac{R_{22}(\vec{e}_1 x_1)}{\langle \tilde{u}_2^2 \rangle}, \\ &R_{22}(\vec{e}_1 x_1) = R_{33}(\vec{e}_1 x_1), \\ &R_{ij}(\vec{e}_1 x_1) = 0, \text{ for } i \neq j. \end{aligned} \tag{4.54}$$

With the incompressibility condition $\partial_j R_{ij} = 0$, we can derive the following relation from (4.53):

$$g(x) = f(x) + \frac{1}{2} x \partial_x f(x). \tag{4.55}$$

For deriving (4.55), it has been used that

$$\partial_j x = \partial_j (x_i^2)^{1/2} = \frac{x_i}{x} \tag{4.56}$$

and

$$\partial_j \left(\frac{x_i x_j}{x^2} \right) = \frac{2x_i}{x^2}. \tag{4.57}$$

With (4.54), the spectra $E_{11}(K_1)$ and $E_{22}(K_1)$ may be expressed as

$$E_{11}(K_1) = \frac{2}{\pi} \langle \tilde{u}_1^2 \rangle \int_0^\infty f(x_1) \cos(K_1 x_1) \, \mathrm{d}x_1 \tag{4.58}$$

and

$$E_{22}(K_1) = \frac{2}{\pi} \langle \tilde{u}_2^2 \rangle \int_0^\infty g(x_1) \cos(K_1 x_1) \, \mathrm{d}x_1. \tag{4.59}$$

In the same way as the integral time scale $\mathcal{T}$ defined in (4.5), the longitudinal and transverse integral length scales l_{11} and l_{22}, respectively, can be defined:

$$l_{11} = \int_0^\infty f(x) \, \mathrm{d}x, \tag{4.60}$$

and

$$l_{22} = \int_0^\infty g(x) \, \mathrm{d}x. \tag{4.61}$$

The question now is, how the one dimensional spectra are related to the three-dimensional spectrum $E(k)$ from (4.16). For finding this, we need to do some further calculations. Let us first reformulate (4.14) by setting $i = j = 1$ and $\vec{x} = \vec{e}_1 x_1$:

$$\begin{aligned} R_{11}(\vec{e}_1 x_1) &= \iiint_{-\infty}^{\infty} E_{11} e^{iK_1 x_1}(\vec{K}) \, \mathrm{d}K_1 \, \mathrm{d}K_2 \, \mathrm{d}K_3 \\ &= \int_{-\infty}^{\infty} \left(\iint_{-\infty}^{\infty} E_{11}(\vec{K}) \, \mathrm{d}K_2 \, \mathrm{d}K_3 \right) e^{iK_1 x_1} \, \mathrm{d}K_1 \\ &= \int_0^\infty E_{11}(K_1) \cos(K_1 x_1) \mathrm{d}K_1 \\ &= \int_{-\infty}^{\infty} \frac{1}{2} E_{11}(K_1) e^{iK_1 x_1} \mathrm{d}K_1, \end{aligned} \tag{4.62}$$

(where we used (4.50)) such that we can conclude

$$E_{11}(K_1) = 2 \iint_{-\infty}^{\infty} E_{11}(\vec{K}) \, \mathrm{d}K_2 \, \mathrm{d}K_3. \tag{4.63}$$

In analogy, we conclude

$$E_{22}(K_1) = 2 \iint_{-\infty}^{\infty} E_{22}(\vec{K})\, \mathrm{d}K_2\, \mathrm{d}K_3. \tag{4.64}$$

We further need to consider the three-dimensional spectrum $E_{ij}(\vec{K})$ under isotropic conditions. Since for that case the spectrum can only depend on the wave number vector $\vec{K}$ itself (see also (4.53)), it must be of the following form:

$$E_{ij}(\vec{K}) = A(K)\delta_{ij} + B(K)K_i K_j. \tag{4.65}$$

It can further be generally shown that the incompressibility condition can be spectrally formulated as

$$K_i E_{ij}(\vec{K}) = 0. \tag{4.66}$$

Therefore, with (4.65) and (4.66), we obtain

$$B(K) = -\frac{A(K)}{K^2}, \tag{4.67}$$

such that

$$E_{ij}(\vec{K}) = A(K)\left(\delta_{ij} - \frac{K_i K_j}{K^2}\right). \tag{4.68}$$

With (4.17), we conclude

$$\begin{aligned}
E(K) &= \oint \frac{1}{2} E_{ii}(\vec{K}) \mathrm{d}\mathcal{S}(K) \\
&= \oint \frac{1}{2} A(K)\left(\delta_{ii} - \frac{K_i K_i}{K^2}\right) \mathrm{d}\mathcal{S}(K) \\
&= \oint \frac{1}{2} A(K)\,(3-1)\, \mathrm{d}\mathcal{S}(K) \\
&= A(K) \oint \mathrm{d}\mathcal{S}(K) \\
&= A(K) 4\pi K^2,
\end{aligned} \tag{4.69}$$

such that

$$A(K) = \frac{E(K)}{4\pi K^2} \tag{4.70}$$

and thus

$$E_{ij}(\vec{K}) = \frac{E(K)}{4\pi K^2}\left(\delta_{ij} - \frac{K_i K_j}{K^2}\right). \tag{4.71}$$

Therefore, with (4.63), we obtain

$$E_{11}(K_1) = \iint_{-\infty}^{\infty} \frac{E(K)}{2\pi K^2}\left(1 - \frac{K_1^2}{K^2}\right) \mathrm{d}K_2\, \mathrm{d}K_3. \tag{4.72}$$

(4.72) is an integration over the entire K_2-K_3 plane, for fixed K_1. Let us define

$$K_r = \sqrt{K_2^2 + K_3^2} \quad \Rightarrow \quad K_r^2 = K^2 - K_1^2 \tag{4.73}$$

(with $K = \sqrt{K_1^2 + K_2^2 + K_3^3}$) as the radial wave number in the K_2-K_3 plane. Then, since the integrand is radially symmetric, (4.72) can be expressed as

$$E_{11}(K_1) = \int_0^\infty \frac{E(K)}{2\pi K^2}\left(1 - \frac{K_1^2}{K^2}\right) 2\pi K_r \mathrm{d}K_r. \tag{4.74}$$

With (4.73), we see that $K_r \mathrm{d}K_r = K\mathrm{d}K$, and $K(K_r = 0) = K_1$ such that (4.74) can be reformulated into

$$E_{11}(K_1) = \int_{K_1}^\infty \frac{E(K)}{K}\left(1 - \frac{K_1^2}{K^2}\right) \mathrm{d}K. \tag{4.75}$$

By taking the cosine Fourier transform of (4.55), see (4.58) and (4.59), we obtain for isotropic turbulence with $\langle \tilde{u}_1^2 \rangle = \langle \tilde{u}_2^2 \rangle$ a relationship between the one-dimensional spectra $E_{11}(K_1)$ and $E_{22}(K_1)$:

$$E_{22}(K_1) = E_{11}(K_1) + \frac{1}{\pi}\langle \tilde{u}_1^2 \rangle \int_0^\infty x_1 \partial_x f(x_1)\cos(k_1 x_1)\,\mathrm{d}x_1. \tag{4.76}$$

With general integration by parts with any a and b, we obtain:

$$\begin{aligned} \int_a^b x\partial_x f(x)\cos(kx)\,\mathrm{d}x \quad &= \quad bf(b)\cos(kb) - af(a)\cos(ka) \\ &\quad - \int_a^b f(x)\cos(kx)\,\mathrm{d}x \\ &\quad + \int_a^b xkf(x)\sin(kx)\,\mathrm{d}x, \end{aligned} \tag{4.77}$$

such that with $f(x) \to 0$ for $x \to \infty$ we obtain from (4.76) the following equation:

$$E_{22}(K_1) = \frac{1}{2}E_{11}(K_1) + \frac{1}{\pi}\langle \tilde{u}_1^2 \rangle \int_0^\infty x_1 K_1 \partial_x f(x_1)\sin(k_1 x_1)\,\mathrm{d}x_1. \tag{4.78}$$

On the other hand we directly obtain the following identity from (4.58):

$$\frac{\mathrm{d}E_{11}(K_1)}{\mathrm{d}K_1} = -\frac{2}{\pi}\langle \tilde{u}_1^2 \rangle \int_0^\infty x_1 K_1 \partial_x f(x_1)\sin(k_1 x_1)\,\mathrm{d}x_1, \tag{4.79}$$

such that combination of (4.78) with (4.79) leads to

$$E_{22}(K_1) = \frac{1}{2}\left(E_{11}(K_1) - K_1 \frac{\mathrm{d}E_{11}(K_1)}{\mathrm{d}K_1}\right). \tag{4.80}$$

With (4.42) and (4.75), we finally obtain for the longitudinal spectrum

$$E_{11}(K_1) = \frac{18}{55} K_0 \varepsilon^{2/3} K_1^{-5/3} \tag{4.81}$$

and for the transverse spectrum, we obtain from this by means of (4.80):

$$E_{22}(K_1) = \frac{24}{55} K_0 \varepsilon^{2/3} K_1^{-5/3}. \tag{4.82}$$