Chapter 5

Statistical turbulence models

The closure problem of turbulence has already been introduced in section 3.3. Turbulence modelling is a way to close the mathematical equations by means of introducing empirical parameterisations. There, the Friedmann-Keller series (see Keller and Friedmann [1924]) is cut off at a certain level and all remaining high-order terms are then expressed by known terms of lower order. Depending on which level the system of equations is closed, we talk about 1. order, 2. order or even higher order closures:

1. order closures: all second-order correlators are described by mean flow gradient.
2. order closures: the second-order correlators are calculated by means of the dynamic equations and all higher-order correlators are described by mean flow gradients or gradients of second-order correlators. However, often, this definition of 2. order closure is not used in a strict sense, since further simplifications are allowed. The pressure-strain correlators (see e.g. eq. (3.29)) which are also second-order correlators may be parameterised by other correlators as well. When the whole left hand side (the transport part of the dynamic equation) is neglected such that an algebraic equation remains, the closure is still called a 2. order closure, see e.g. ?.
3. order closures are defined accordingly.

5.1 Eddy viscosity principle

One of the basic principles of turbulence modelling is the so-called eddy viscosity assumption first formulated by Boussinesq. If we assume a highly idealised flow in x-direction with \( \bar{u} = \bar{u}(z) \), then the total stress in the Reynolds-averaged Navier-Stokes equations (3.17) may be formulated as:

\[
\tau = \tau_m + \tau_t = \rho_0 (\nu \partial_z \bar{u} - \langle \tilde{u}\tilde{v} \rangle) .
\]

With the simple assumption that the Reynolds stress may be modelled in the same way as the viscous stress, the following relation is obtained:

\[
-\langle \tilde{u}\tilde{v} \rangle = \nu_t \partial_z \bar{u},
\]

with the so-called eddy viscosity \( \nu_t \), which is introduced as a new unknown. Combining (5.1) and (5.2), we finally obtain:

\[
\tau = \tau_m + \tau_t = \rho_0 (\nu + \nu_t) \partial_z \bar{u}.
\]

The eddy viscosity assumption (5.2) is motivated by means of the sketch in figure 5.1. Assuming a velocity profile linearly increasing with \( z \), i.e. the mean velocity gradient \( \partial_z \bar{u} \) is positive and constant, then a negative vertical velocity deviation \( \tilde{w} < 0 \) would statistically move faster water down into slower
Figure 5.1: Sketch illustrating why the Reynolds stress $\langle \tilde{u}\tilde{w} \rangle$ is proportional to the negative mean velocity gradient $\partial_z \bar{u}$.

water such that for this case we expect $\tilde{u} > 0$. The opposite would be expected for a positive vertical velocity deviation $\tilde{w} > 0$, which would typically lead to $\tilde{u} < 0$. Thus statistically, for this case, we would expect a negative Reynolds stress $\langle \tilde{u}\tilde{w} \rangle$. Furthermore, the absolute value of the Reynolds stress would be larger for a steeper mean velocity gradient. Thus, the eddy viscosity assumption (5.2) makes sense physically.

Unlike the molecular viscosity which varies only slightly with temperature and is thus usually considered as a material constant, the eddy viscosity is highly variable in time and space. It can be up to some orders of magnitude larger than the molecular viscosity. A generalisation of (5.2) to non-idealised three-dimensional flows would be of the following form (considering symmetry of the second-order correlators):

$$\langle \tilde{v}_i \tilde{v}_j \rangle = \nu_t \left( \partial_i \bar{v}_j + \partial_j \bar{v}_i \right) - \frac{2}{3} k \delta_{ij}.$$  \hspace{1cm} (5.4)

The term is necessary in order to allow proper application of the Einstein summation rule for $i = j$.

The eddy diffusivity is defined analogously:

$$-\langle \tilde{v}_i \tilde{c} \rangle = \nu'_t \partial_i \bar{c}$$ \hspace{1cm} (5.5)

with the eddy diffusivity $\nu'_t$ and the scalar quantity $c$ which may be e.g. potential temperature $\theta$ or salinity $S$. The ratio of the eddy viscosity to the eddy diffusivity is the turbulent Prandtl number

$$P^t_r = \frac{\nu_t}{\nu'_t}.$$ \hspace{1cm} (5.6)

which is of the order of unity and increases with stratification. It should be noted that there are various simplifications included in the eddy viscosity approach presented above:

- Since the flow is anisotropic on the scales of large eddies, the eddy viscosity should be direction depending, such that it should be expressed by means of a tensor rather than by means of a scalar. Since vertical mixing plays in ocean models a much more important role than horizontal mixing (which is mostly an consequence of vertical mixing processes), turbulence modelling concentrates on the vertical eddy viscosity and diffusivity.

- Turbulence has a memory which is neglected when the eddy viscosity is directly related to mean-flow quantities. Thus dynamic turbulence models (see section 5.5) are more realistic than fully algebraic approaches (see sections 5.2 and 5.4).
The problem of calculating the eddy viscosity and the eddy diffusivity will be discussed in the following sections 5.2 – 5.5.

5.2 Mixing length approach

Prandtl and Kolmogorov suggested to parameterise the eddy viscosity and diffusivity in a way motivated by molecular diffusion in ideal gases: as the product of a characteristic length and a characteristic velocity:

$$\nu_t = l \cdot v.$$  \hspace{1cm} (5.7)

In Prandtl’s mixing length approach, the characteristic velocity is perpendicular to the direction of the mean flow and is assumed to be the product of the characteristic length and the gradient of the mean flow:

$$v = l \partial_z \bar{u}.$$  \hspace{1cm} (5.8)

such that

$$\nu_t = l^2 \partial_z \bar{u}.$$  \hspace{1cm} (5.9)

Thus, the Reynolds stress may be formulated as

$$\tau_t = -\rho_0 (\bar{u} \bar{w}) = \rho_0 \nu_t \partial_z \bar{u} = \rho_0 l^2 \partial_z \bar{u} \partial_z \bar{u},$$  \hspace{1cm} (5.10)

where the module of the mean shear is introduced in order to have consistent signs for the Reynolds stress. Thus, the problem is reduced to determining the mixing length \( l \). This is relatively simple in simple geometries such as for flows over a solid surface.

5.3 Bottom boundary layer

Let us assume a fully developed, pressure gradient driven flow over a smooth bed which is irrotational, directed along the \( x \)-axis, has no gradients along the \( x \)-axis (except the pressure gradient) and a constant density \( \rho = \rho_0 \). Then the Reynolds equations (3.17) would simplify to a balance of the pressure gradient with the vertical stresses:

$$\partial_z (\bar{u} \partial_z \bar{u} + \langle \bar{u} \bar{w} \rangle) = -\frac{1}{\rho_0} \partial_x \bar{p}.$$  \hspace{1cm} (5.11)

With defining the total stress as

$$\tau = -\rho_0 (\bar{u} \partial_z \bar{u} + \langle \bar{u} \bar{w} \rangle),$$  \hspace{1cm} (5.12)

(the first term is the viscous stress and the second term is the Reynolds stress) we obtain the following balance

$$\partial_z \tau = \partial_x \bar{p}.$$  \hspace{1cm} (5.13)

With \( \partial_x \bar{p} \) being a constant, \( \partial_z \tau \), such that the vertical profile of the total stress is linear.

With (5.11), it is obvious that \( \bar{u} \) for a smooth bed (with no length scale given by any roughness elements) can only be a function of density \( \rho_0 \), bottom stress \( \tau_b \), viscosity and distance from the bed, \( z \):

$$\bar{u} = f(\rho_0, \tau_b, \nu, z).$$  \hspace{1cm} (5.14)

Since \( \rho_0 \) and \( \tau_b \) are the only quantities containing density, the ratio

$$u_* = \sqrt{\frac{\tau_b}{\rho_0}}$$  \hspace{1cm} (5.15)
which is the so-called friction velocity, is of great importance. Thus, \((5.14)\) may be reformulated as
\[
\bar{u} = g(u_*, \nu, z). \tag{5.16}
\]
Since there are only two possible pairs of non-dimensional parameters, dimensional analysis gives the relation
\[
\frac{\bar{u}}{u_*} = h \left( \frac{u_* z}{\nu} \right) = h(z_+) \tag{5.17}
\]
with \(z_+\) being defined as the non-dimensionalised distance from the wall (or bed). \((5.17)\) is called the universal law of the wall.

The **viscous sublayer** is defined as the part of the flow close to the bed where the Reynolds stress is negligible with respect to the viscous stress. This layer is assumed to be so thin that the stress can be considered as constant within this layer. Thus, \((5.12)\) simplifies to
\[
\frac{\tau}{\rho_0} = \frac{\tau_0}{\rho_0} = \nu \partial_z \bar{u}, \tag{5.18}
\]
which after integration and application of the no-slip boundary condition \(\bar{u}(0) = 0\) gives:
\[
\bar{u} = \frac{z \tau_0}{\rho_0 \nu}, \tag{5.19}
\]

or
\[
\frac{\bar{u}}{u_*} = \frac{z u_*}{\nu} = z_+. \tag{5.20}
\]

Experiments show that the viscous sublayer extends to \(0 \leq z_+ \leq 5\).
Further away from the wall where viscous stresses are negligible compared to Reynolds stresses, the turbulent length scale \(l\) should be proportional to the distance from the wall. This can be nicely studied for a so-called constant stress layer as obtained by means of a Couette flow near one of the walls. For \(\bar{u} > 0\) and \(\partial_z \bar{u} > 0\), we obtain:
\[
\frac{\tau}{\rho_0} = u_*^2 = -\langle \bar{u} \bar{v} \rangle = \text{const.} \tag{5.21}
\]
According to \((5.10)\), we obtain
\[
l^2 (\partial_z \bar{u})^2 = u_*^2 \tag{5.22}
\]
or
\[
l \partial_z \bar{u} = u_*. \tag{5.23}
\]
Because of \(l \propto z\) we formulate
\[
l = \kappa z \tag{5.24}
\]
with an unknown proportionality factor \(\kappa\) and obtain
\[
\partial_z \bar{u} = \frac{u_*}{\kappa z}, \tag{5.25}
\]
or, equivalently,
\[
\partial_{z_+} \bar{u} = \frac{u_*}{\kappa z_+}, \tag{5.26}
\]
and, after integration:
\[
\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln z_+ + A, \tag{5.27}
\]

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Figure 5.2: Sketch of the universal law of the wall. This figure has been taken from *Kundu and Cohen* [2002].

with the integration constant $A$. This is the logarithmic law of the wall. It has been shown empirically that $A = 5$ and that (5.27) is valid for $z_+ > 30$, see figure 5.2. Experiments further show that $\kappa \approx 0.41$ which is called the von Kármán constant.

Between the viscous sublayer (viscous stresses only) and the logarithmic layer (Reynolds stresses only), a buffer layer exists for $5 < z_+ < 30$.

When searching for the zero velocity in (5.27) one finds $\bar{u} = 0$ for $z_+ = z_+^0 \approx 0.13$, which is clearly outside the validity of the logarithmic law of the wall. Therefore, the physical length at which the velocity of the logarithmic law becomes zero is

$$z_0 = \frac{z_+^0 \nu}{u_*} \approx 0.13 \frac{\nu}{u_*}. \quad (5.28)$$

Reformulation of (5.27) with (5.28) results in

$$\frac{\bar{u}(z)}{u_*} = 1 - \frac{1}{\kappa} \ln \left( \frac{z}{z_0} \right). \quad (5.29)$$

For hydrodynamically rough beds with the length of roughness elements $K_0$, the viscous sublayer is completely submerged in the roughness elements and (5.29) is valid outside the roughness elements with $z_0 \approx K_0/30$.

However, the constant stress assumption, although often made, is unrealistic. Let us assume a stationary channel flow with finite water depth $D$ and a constant pressure gradient $\partial_x \bar{p} < 0$, leading to a positive velocity profile $\bar{u}$. Again neglecting viscous forces, we obtain from (5.11) the following expression for the Reynolds stress:
\[ -\partial_z \left( \frac{\tau_s}{\rho_0} \right) = \partial_z \langle \tilde{u} \tilde{w} \rangle = \frac{1}{\rho_0} |\partial_x \tilde{p}|, \]  
(5.30)

which after vertical integration leads to

\[ -\frac{\tau_s}{\rho_0} + \frac{\tau_b}{\rho_0} = \frac{1}{\rho_0} |\partial_x \tilde{p}| D. \]  
(5.31)

With assuming zero surface stress \((\tau_s = 0)\) and defining the bottom friction velocity as

\[ u_b^* = \sqrt{\frac{\tau_b}{\rho_0}}, \]  
(5.32)

we obtain

\[ \frac{\tau}{\rho_0} = \nu_t \partial_z \tilde{u} = -\langle \tilde{u} \tilde{w} \rangle = \left( 1 - \frac{z}{D} \right) \left( u_b^* \right)^2. \]  
(5.33)

As a consequence of (5.10) and (5.24), the eddy viscosity for the constant stress case was formulated as

\[ \nu_t = z \kappa u^*_b, \]  
(5.34)

which is unrealistic in this limited depth case, since viscosity should also decrease near the surface, such that for this case a parabolic shape of the eddy viscosity would be more realistic:

\[ \nu_t = z \left( 1 - \frac{z}{D} \right) \kappa u^*_b, \]  
(5.35)

with the bottom friction velocity from (5.32). By combining (5.33) and (5.35), we finally obtain

\[ \partial_z \tilde{u} = \frac{u_b^*}{\kappa z}, \]  
(5.36)

which is of the same form as (5.25) and thus leads to the logarithmic law of the wall as well. It should be noted that with (5.9), (5.35) and (5.36) for the pressure gradient driven flow, the length scale has the following shape:

\[ l = \kappa z \sqrt{1 - \frac{z}{D}} \]  
(5.37)

and is thus asymmetric. However, towards the bed it converges to the constant stress formulation of \( l \), which was (5.24).

### 5.4 Turbulence equilibrium

For more complex, stratified situations, it is often exploited that turbulence tends to an equilibrium state, which is given by assuming steady-state conditions and neglect of transport terms in the dynamic equation for turbulent kinetic energy. When applying the eddy viscosity assumption (5.4), we obtain from (3.36) and neglecting horizontal gradients with respect to vertical gradients, we obtain

\[ P + B = \varepsilon \]  
(5.38)

with

\[ P = \nu_t \left( (\partial_z \tilde{u})^2 + (\partial_z \tilde{v})^2 \right), \quad B = -\nu_t' \partial_z b. \]  
(5.39)
The buoyancy production $B$ is here derived from
\[
B = -\frac{g}{\rho_0} \langle \tilde{w} \tilde{\rho} \rangle = \frac{g}{\rho_0} \nu' \partial_z \rho \tag{5.40}
\]
and the definition of the buoyancy $b$:
\[
b = -g \overline{\tilde{\rho}} - \frac{\rho_0}{\rho_0}. \tag{5.41}
\]
When taking $k^{1/2}$ as proportional to the characteristic velocity scale to be inserted into the Kolmogorov-Prandtl relation (5.7), we obtain
\[
\nu_t = c_l k^{1/2}, \tag{5.42}
\]
with the constant of proportionality, $c_l$. When integrating the turbulence spectrum for the inertial subrange given in equation (4.42), from wave number at the mixing length scale, $K_l = 2\pi/l$ to infinity, and assume that that portion of the turbulent kinetic energy is proportional to the turbulent kinetic energy, $k$, then we obtain a relation between $k$, $l$ and $\varepsilon$:
\[
\varepsilon = c_d k^{3/2} l, \tag{5.43}
\]
with the constant of proportionality, $c_d$, and thus an algebraic expression for the turbulent kinetic energy, which does only depend on mean flow properties and the mixing length $l$:
\[
k = \frac{c_l c_d l^2}{l} \left( (\partial_z \bar{u})^2 + (\partial_z \bar{v})^2 + \frac{\partial_z b}{P_T} \right)^{1/2} \tag{5.44}
\]
such that with (5.42) we obtain
\[
\nu_t = \left( \frac{c_l^2}{c_d} \right)^{1/2} \left( \left( \partial_z \bar{u} \right)^2 + \left( \partial_z \bar{v} \right)^2 + \frac{\partial_z b}{P_T} \right)^{1/2}, \tag{5.45}
\]
which is consistent with the mixing length approach (5.9). A simple algebraic approximation for the mixing length $l$ under consideration of stratification would be for example:
\[
l = \left( \frac{1}{\kappa z} \left( 1 - \frac{\varepsilon}{N^2} \right)^{1/2} + \frac{1}{c \sqrt{k^{1/2}} N} \right)^{-1} \tag{5.46}
\]
with the constant $c$, and the Brunt-Väisälä frequency $N$ with $N^2 = \partial_z b$. This approach combines the geometric length scale (5.37) with the definition of the Ozmidov length scale,
\[
l_O = \left( \frac{\varepsilon}{N^3} \right)^{1/2}, \tag{5.47}
\]
which is a limitation of turbulent eddies due to stable stratification, and leads with (5.43) to
\[
l_O \propto \frac{k^{1/2}}{N}, \tag{5.48}
\]
and thus to the form in which it is introduced into (5.46).
5.5 Dynamic turbulence models

5.5.1 Boundary conditions

When dynamic equations are used for calculating turbulent quantities, boundary conditions are needed. These are usually derived from the constant stress law of the wall with

\[ l = \kappa z, \]
\[ \partial_z \bar{u} = \frac{u_\kappa}{\kappa z}, \]
\[ P = \varepsilon, \]
\[ \nu_t \partial_z \bar{u} = u_*^2, \]

i.e. a constant stress layer is assumed in the region where the boundary conditions are located. From combining the first, the second and the fourth equation of (5.49) with (5.42), a boundary condition for \( k \) may be derived:

\[ k = \frac{u_*^2}{c_l^2}. \]

With combining the first, the second and the third equation of (5.49) with (5.42), we obtain

\[ k = \frac{c_l}{c_d} u_*^2, \]

such that \( c_d = c_l^3 \) with the consequence that (5.45) is simplified.

With combining (5.43) with (5.50), we obtain the profile of the dissipation rate \( \varepsilon \) within the logarithmic boundary layer:

\[ \varepsilon = \frac{u_*^3}{\kappa z}. \]

Typically, the bottom boundary conditions are set at the position where the logarithmic velocity profile is zero (this is typically not the real bottom due to the existence of a viscous sublayer), which is at \( z = z_0 \), see equation (5.29), such that the bottom boundary condition for \( \varepsilon \) is

\[ \varepsilon = \frac{u_*^3}{\kappa z_0}. \]

5.5.2 The \( k-\varepsilon \) model

It has already been discussed that the dynamic equation for the turbulent kinetic energy (3.36) is closed on the right hand side, after applying the eddy viscosity assumption. The transports terms on the left hand side however need to be closed, and this is typically done by assuming that these transports are proportional to the gradient of the TKE (the so-called down-gradient assumption). With these approximations, the TKE equation reads as

\[ \partial_t k - \partial_z \left( \nu + \frac{\nu_t}{\sigma_k} \right) \partial_z k = P + B - \varepsilon, \]

with the turbulent Schmidt number for TKE diffusion, \( \sigma_k \).

Also for the dissipation rate, a dynamic equation can be derived from the Reynolds decomposition of the Navier-Stokes equations, which however contains a high number of unclosed terms. Therefore, a common
procedure for the closure of that equation is to assume that its shear production, buoyancy production and dissipation terms on the right hand side are proportional to the equivalent terms in the TKE equation (5.54). The resulting equation is of the following form:

$$\partial_t \varepsilon - \partial_z \left( \left( \nu + \frac{\nu_t}{\sigma_\varepsilon} \right) \partial_z \varepsilon \right) = \frac{\varepsilon}{k} \left( c_1 P + c_3 B - c_2 \varepsilon \right),$$

(5.55)

with the four empirical coefficients $c_1$, $c_2$, $c_3$ and $\sigma_\varepsilon$. The system of equations (5.54) and (5.55) gives the well-known $k-\varepsilon$ model turbulence closure model.

With combining (5.42) and (5.43), we obtain

$$\nu_t = c_4 \frac{\kappa^2}{\varepsilon}.$$  

(5.56)

With a formulation for the turbulent Prandtl number $P_{tr}$, the $k-\varepsilon$ model is closed:

$$P_{tr} = P_{tr0} \exp \left( - \frac{R_i}{P_{tr0} R_i^\infty} \right) + \frac{R_i}{R_i^\infty}$$

(5.57)

with the neutral turbulent Prandtl number $P_{tr0} = 0.74$ and $R_i^\infty = 0.25$. In (5.57), $R_i = N^2/M^2$ with the shear squared $M^2 = (\partial_z \bar{u})^2 + (\partial_z \bar{v})^2$ is the gradient Richardson number.