Assignment Nr. 1

Please, send your solutions via e-mail to evridiki.chrysagi@io-warnemuende.de
due **Friday, 23st April (08:00 am)**

General remarks: Please send in a) the programmed code b) the plots and/or necessary comments in PDF-file until due date. Make sure that everything is appended, readable and in English. This ensures that your solutions can be be corrected prior to the next assignment sheet. As programming language I would recommend you to use an open source language, e.g. MatLab, Python, FORTRAN, C, C++, R. If any questions occur, do not hesitate to contact me via e-mail or phone 0381-5197-3489.

Problem 1 (22 points)

The partial differential equation discussed in this assignment is the one-dimensional advection equation,

$$\frac{\partial \psi}{\partial t} + c\frac{\partial \psi}{\partial x} = 0 \ , \tag{1}$$

where $\psi = \psi(x,t)$ is a function varying in the spatial component x and in time t, c denotes the constant advection speed. $\psi(x,t)$ is defined for $0 \le x \le L$ and $0 \le t \le T$. Initial conditions are established by defining $\psi(x,0) = \psi_0(x)$. Boundary conditions are assumed to be periodic for simplicity, i.e. $\psi(0,t) = \psi(L,t)$.
To obtain a discrete approximation to $\psi(x,t)$ let the spatial domain be discretised by M intervals with spacing $h_i = x_i - x_{i-1}$ with $x_{i-1} < x_i$ for $1 \le i \le M$, and $x_0 = 0$ and $x_M = L$. The time is discretised by N equidistant intervals with $t_j = j\Delta t$ for $0 \le j \le N$ and $N\Delta t = T$. The approximate discrete solution defined on the grid points (x_i, t_j) for $0 \le i \le M$ and $0 \le j \le N$ will be denoted by ϕ_i^j.
(Note that, in contrast to the notation used here, in some programming languages the index of the first grid point starts at 1 by default, e.g. in FORTRAN and MATLAB.)

(a) Show that for periodic boundary conditions, an analytical solution to (1) is given by

$$\psi(x,t) = \psi_0(\xi), \quad \xi = \left(\frac{x-ct}{L} - \left\lfloor \frac{x-ct}{L} \right\rfloor\right) L \ , \tag{2}$$

where $\lfloor \frac{x-ct}{L} \rfloor$ is the largest integer smaller than $\frac{x-ct}{L}$ (2 points).

(b) Develope a numerical algorithm that provides a numerical solution to (1) by using a programming language of your choice. To this end, use the numerical scheme specified by

$$\frac{\phi_i^{j+1} - \phi_i^j}{\Delta t} + c\,\frac{\phi_i^j - \phi_{i-1}^j}{h_i} = 0 \ , \tag{3}$$

for $i = 0, \ldots, M$ and $j = 0, \ldots, N$ and $c \ge 0$. This scheme is known as the *up-wind*, *up-stream* or *donor cell* scheme.

Choose $c = 1$ and use periodic boundary conditions, $\phi_0^j = \phi_M^j$, as well as the top-hat inital condition

$$\psi_0(x) = \begin{cases} 1, & \text{for } 0.4L \leq x \leq 0.6L, \\ \\ 0, & \text{else.} \end{cases} \tag{4}$$

Discretise the domain using the parameters $L = 100$, $T = 300$, $M = 100$, and $N = 600$, and a constant grid spacing, h_i.

Plot the analytical solution, $\psi(x, t)$, and the numerical approximation, ϕ_i^j, at times $t = 100$, $t = 200$, $t = 300$. Compare this to the numerical solution for the same problem with $M = 1000$, and $N = 6000$. How and why does the numerical solution differ from the analytical one? Which physical process would yield a similar result as the numerical solution? (20 points).