Assignment Nr. 2

due **Monday, 07th May**
Please, send your solutions via e-mail to evridiki.chrysagi@io-warnemuende.de

Problem 1 (7 points)

The task in this problem is to derive a second-order accurate, one-sided finite difference expression for the derivative of the function $f(x)$ at the point x_0.

(a) By use of Taylor-series expansions of $f(x_0 + \Delta x)$ and $f(x_0 + 2\Delta x)$ around x_0, determine the coefficients a, b, and c in the finite difference approximation of the derivative

$$f'(x_0) = af(x_0) + bf(x_0 + \Delta x) + cf(x_0 + 2\Delta x) + \mathcal{O}((\Delta x)^2) \; , \tag{1}$$

such that the scheme is second-order accurate as indicated. (3 points)

(b) For $f(x) = \sin(x)$, derive approximations for $f'(x_0 = \pi/4)$ for the one-sided difference scheme,

$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + \mathcal{O}(\Delta x) \; , \tag{2}$$

for the central difference scheme,

$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} + \mathcal{O}((\Delta x)^2) \; , \tag{3}$$

and for the one-sided three-point scheme in (1).
Plot the relative discretisation error,

$$e = \frac{|f'(x_0) - f'_h(x_0)|}{f'(x_0)} \; , \tag{4}$$

where f'_h denotes the discrete approximation, respectively, as a function of the grid spacing, $\Delta x = \pi \, 2^{-i}$, for $i = 1, \ldots, 10$. Use a doubly-logarithmic scale, and verify that the slopes of the curves correspond to the power-laws for the discretisation errors shown in (1), (2), and (3). (4 points)

Problem 2 (15 points)

Consider the one-dimensional diffusion equation,

$$\frac{\partial \psi}{\partial t} = \nu \, \frac{\partial^2 \psi}{\partial x^2} \; , \tag{5}$$

with constant diffusivity, ν, subject to the initial condition

$$\psi_0(x) = \cos\left(2\pi \frac{x}{L}\right) \; , \tag{6}$$

where L denotes the length of the periodic domain.

Let a finite-difference approximation to (5) be given by

$$\frac{\phi_i^{j+1} - \phi_i^j}{\Delta t} - \nu \frac{\phi_{i+1}^j - 2\phi_i^j + \phi_{i-1}^j}{(\Delta x)^2} = 0 \quad . \tag{7}$$

(a) Using the one-sided and the central difference scheme show that (7) is consistent with (5). What is the accuracy of this finite difference approximation in Δt and Δx? (2 points)

(b) Show analytically that the exact solution of (5) with initial condition (6) is given by

$$\psi(x,t) = \cos\left(2\pi \frac{x}{L}\right) \exp\left(-\frac{4\pi^2\nu}{L^2}t\right) \quad . \tag{8}$$

(3 points)

(c) Implement the discretised form of the diffusion equation, (7), and the initial condition, (6), into the software developed by you during the first assignment. Comparing the exact and numerical solution for $\nu = 0.1$ and $L = 100$ at times $t = 1000$ and $t = 5000$ in a plot. Use a time step of $\Delta t = 1$, and resolve the spatial domain with $M = 100$ intervals. (4 points)

(d) Show that for the scheme (7) the amplification factor, A_l, of the Fourier mode with wave number k_l is given by

$$A_l = 2D(\cos(k_l \Delta x) - 1) + 1 \ , \tag{9}$$

where

$$D = \nu \frac{\Delta t}{(\Delta x)^2} \tag{10}$$

has been defined for convenience. For $D = 0.1, 0.5$, and 1.0, plot A_l for $0 \leq k_l \leq k_{\max}$, where $k_{\max}$ is the maximum wave number that can be resolved on the numerical grid. How do you derive the value of $k_{\max}$? (3 points)

(e) Show that the stability condition $|A_l| \leq 1$ (i.e. no Fourier mode is allowed to grow) requires $D \leq 1/2$. Which is the most unstable Fourier mode, i.e. for which value of $k_l \Delta x$ do you expect the largest amplification factor if the stability criterion is violated? (in other words: which Fourier mode would be the first to be visible when the simulation becomes unstable)? Conduct a numerical run and show that this mode indeed dominates the numerical solution if you violate the stability condition and compute for a long enough period. (3 points)