Physics and Numerics in GETM

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Contents

• Physics
  • Standard physics for transport
  • Complex statistical mixing schemes
  • Drying & flooding

• Numerics
  • Mode splitting
  • Horizontal grids
  • Vertical grids
  • High-order monotone advection schemes
  • Various pressure gradient schemes
Model Requirements I

• Algorithm for drying and flooding for simulating Wadden Sea dynamics.

• Bottom-fitted coordinates for better representation of near-bottom flows.

• Surface-fitted coordinates for high vertical near-surface resolution and with large tidal amplitude.

• General vertical coordinates for better fitting of the model grid to the internal flow structures.
Model Requirements II

- **Curvilinear horizontal coordinates** for better representation of complex bathymetry and higher resolution of narrow regions without nesting.

- **Monotone high-order advection schemes** for better representation of fronts and stratification.

- **High-order turbulence models** for good representation of vertical mixing.
Physics

In GETM, standard physical laws for transport in fairly shallow natural waters (small aspect ratio) are implemented:

- 3D primitive equations
- hydrostatic approximation
- Boussinesq approximation
- Free surface

Extensions (non-Boussinesq, non-hydrostatic) seem feasible.
Non-standard are the high-order turbulence models and the simplifications for drying & flooding.
Physics - Turbulence Modelling

GETM uses the turbulence module of the water column (1D) model GOTM, which includes the following parameterisations:

- Zero-equation models (algebraic TKE)
- One-equation models (algebraic length scales)
- Two-equation models as work horses such as:
  - $k$-$\varepsilon$ model
  - Mellor-Yamada model
  - Generic two-equation model (e.g. $k$-$\omega$)
- Various algebraic second moment closures
- Non-local KPP model coming soon
$k$-$\varepsilon$ model I

Turbulent Fluxes (velocity & temperature):

$$\langle \tilde{u}\tilde{w} \rangle = -\nu_t \partial_z \bar{u}, \quad \langle \tilde{w}\tilde{T} \rangle = -\nu'_t \partial_z \bar{T}$$

Eddy Viscosity / Eddy Diffusivity:

$$\nu_t = c\mu \frac{k^2}{\varepsilon}, \quad \nu'_t = c'_\mu \frac{k^2}{\varepsilon}.$$  

$k$: Turbulent kinetic energy (TKE) [J/kg]  
$\varepsilon$: Dissipation of TKE [W/kg]
**$k$-$\varepsilon$ model II**

$k$-$\varepsilon$ model ([Launder and Spalding [1972]]):

\[
\partial_t k - \partial_z \left( \frac{\nu_t}{\sigma_k} \partial_z k \right) = P + B - \varepsilon,
\]

\[
\partial_t \varepsilon - \partial_z \left( \frac{\nu_t}{\sigma_\varepsilon} \partial_z \varepsilon \right) = \frac{\varepsilon}{k} \left( c_{\varepsilon 1} P + c_{\varepsilon 3} B - c_{\varepsilon 2} \varepsilon \right).
\]

*P:* Shear production of TKE [W/kg]

*B:* Buoyancy production [W/kg]
Total equilibrium \((k-\varepsilon)\)

\[
\dot{k} = \dot{\varepsilon} \implies R_i = \frac{-g \frac{\partial z \rho}{\rho_0}}{(\partial_z u)^2 + (\partial_z v)^2} = R_{i}^{st} = \frac{c_{\mu}}{c'_{\mu}} \cdot \frac{c_{2\varepsilon} - c_{1\varepsilon}}{c_{2\varepsilon} - c_{3\varepsilon}}.
\]

\(R_{i}^{st} \approx 0.25\): Steady-state Richardson number.

Burchard & Bolding [2001]
Kato-Phillips experiment

Wind-induced mixed-layer depth (MLD)

MLD for Canuto et al. version A

- Empirical
- $Ri_{st} = 0.2$
- $Ri_{st} = 0.3$
- $Ri_{st} = 0.4$
- $Ri_{st} = 0.6$
- $Ri_{st} = 0.8$

Burchard & Bolding [2001]
GOTM: Liverpool Bay

SST from space and location of station (●)

Courtesy to School of Ocean Sciences, UBW, Wales
GOTM: Liverpool Bay

Section of Temperature and Salinity

Rippeth, Fisher, Simpson [2001]
GOTM: Liverpool Bay

Observed and simulated temperature and salinity

Simpson, Burchard, Fisher, Rippeth [2002]
GOTM: Liverpool Bay

Observed and simulated current velocity

Simpson, Burchard, Fisher, Rippeth [2002]
GOTM: Liverpool Bay

Observed and simulated dissipation rates

Simpson, Burchard, Fisher, Rippeth [2002]
Drying: Physical mechanism

Momentum equation:

\[ \partial_t u + \partial_z (uw) - \partial_z ((\nu_t + \nu) \partial_z u) \]

\[ + \alpha \left( \partial_x (u^2) + \partial_y (uv) - \partial_x (2A_h^M \partial_x u) - \partial_y (A_h^M (\partial_y u + \partial_x v)) \right) \]

\[ - f v - \int_z^\zeta \partial_x b \, dz' \right) = -g \partial_x \zeta, \]

\[ \alpha = \min \left\{ 1, \frac{D - D_{\text{min}}}{D_{\text{crit}} - D_{\text{min}}} \right\}, \quad D_{\text{min}} = 2\text{cm}, \quad D_{\text{min}} = 10\text{cm}. \]
Drying: Numerical mechanism

\[-H_{i,j} + D_{\text{min}} \]

\[\zeta_{i,j} \]

\[\tilde{\zeta}_{i+1,j} \]

\[\zeta_{i+1,j} \]

\[-H_{i+1,j} \]

--- Virtual sea surface elevation

--- Actual sea surface elevation

--- Bathymetry approximation
Drying: Sylt-Rømø-Bight I
Eddy viscosity $\nu_t$ along cross-section during high water $\nu_t/(m^2s^{-1})$

Eddy viscosity $\nu_t$ along cross-section during low water $\nu_t/(m^2s^{-1})$
Drying: Sylt-Rømø-Bight III

- $u$, velocity component in point 2
- Eddy viscosity $\nu_t$ in point 2
- Turbulent kinetic energy $k$ in point 2
- Turbulent dissipation rate $\varepsilon$ in point 2

1. GETM Users Workshop, Båring Hojskole, Denmark, June 6-8, 2004 – p. 20/37
Drying: East Frisia

Vertically Integrated Velocity

0.25 tidal period

0.5 tidal period

0.75 tidal period

1 tidal period

Stanov et al. [2002]
Mode splitting

GETM time stepping is based on conservative mode splitting. Fast time step for external (vertically-integrated mode), slow time stepping for internal (vertically-resolved) mode, necessity of mode coupling terms.
General vertical coords.

\( S \)-equation in Cartesian coordinates:

\[
\partial_t^* S + \partial_x^*(uS) + \partial_y^*(vS) + \partial_z^*(wS) - \partial_z^*(\nu'_t \partial_z^* S) = 0. \tag{0}
\]

Coordinate transformation:

\[
\gamma = \gamma(t^*, x^*, y^*, z) \iff z = z(t, x, y, \gamma). \tag{1}
\]

Jacobian of the transformation:

\[
J := \partial_\gamma z = (\partial_z^* \gamma)^{-1}. \tag{2}
\]

\( S \)-equation in transformed coordinates:

\[
\partial_t (JS') + \partial_x (JuS) + \partial_y (JvS) + \partial_\gamma (\tilde{w}S) - \partial_\gamma \left( \frac{\nu'_t}{J} \partial_\gamma S \right) = 0. \tag{3}
\]
General vertical coords.

The same the discrete vertical layer distribution may be obtained

by equidistantly discretising the transformed equations

or

by discretising the equations in Cartesian coordinates by means of layers with moving interfaces using kinematic boundary conditions.

The result is the same: Layers which are basically newly distributed after each time step, guaranteeing mass conservation.
General vertical coords.

Cross-section through Dogger Bank area with various coordinate transformations:
Adaptive vertical coordinates

FLEX’76 simulation (water column in Northern North Sea).

Shear-squared (left) and buoyancy frequency (right)

Burchard and Beckers [2004]
Adaptive vertical coordinates

FLEX’76 simulation (water column in Northern North Sea).

Layer interface evolution for $N = 10, 20, 40$ and $80$ layers

Burchard and Beckers [2004]
Adaptive vertical coordinates

Burchard and Beckers [2004]
Adaptive vertical coordinates

Internal seiche with fixed grid

Y (meters): 1845
TIME: 02-JAN-2000 04:27
DATA SET: lock.3d.fixed.nc

[Diagram showing adaptive vertical coordinates and internal seiche with fixed grid]
Adaptive vertical coordinates

Internal seiche with adaptive grid refining at gradients
Adaptive vertical coordinates

Internal seiche with semi-Lagranian adaptive grid
Advection schemes

GETM has implemented various different advection schemes for tracers and momentum (turbulence advection under development).

- One-dimensional schemes are used in directional-split mode (Pietrzak 1998).
- These schemes are e.g. First-order upstream, ULTIMATE QUICKEST and the TVD-schemes Superbee, MUSCL and $P_2$-PDM.
- Iteration of vertical advection (CFL-criterium).
- As 2D-horizontal schemes we have first-order upstream and FCT, which may be combined with vertical 1D scheme.
2D test case: $P_2$ split scheme

Cube resulting after one solid-body rotation with $\Delta x = \Delta y = 1$ m and a Courant number of $c = 0.5$. Left: unlimited $P_2$ scheme; right: limited $P_2$-PDM scheme

TVD-Verfahren führen zu monotonen Advektionsverfahren höherer Ordnung.
Freshwater eddy I

Left: surface salinity and current vectors; right: bottom current vectors. Momentum advection: multidimensional upwind scheme; salinity advection: TVD-Superbee directional-split scheme.
Freshwater eddy II

Left: surface salinity and current vectors; right: bottom current vectors. Momentum advection: momentum and salinity advection: TVD-Superbee directional-split scheme.
Pressure gradient problem

When sloping coordinate surfaces intersect with isopycnal surfaces, truncation errors occur due to the balance of two large terms:

\[ \frac{1}{2} (h_{i,j,k} + h_{i,j,k+1}) (m \partial_x^* b)_{i,j,k} \]

\[ \approx \frac{1}{2} (h_{i,j,k}^u + h_{i,j,k+1}^u) \left( \frac{1}{2} (b_{i+1,j,k+1} + b_{i+1,j,k}) - \frac{1}{2} (b_{i,j,k+1} + b_{i,j,k}) \right) \frac{\Delta x_{i,j}^u}{ \Delta x_{i,j}} \]

\[ - (\partial_x z_k)_{i,j,k}^x \left( \frac{1}{2} (b_{i+1,j,k+1} + b_{i,j,k+1}) - \frac{1}{2} (b_{i+1,j,k} + b_{i,j,k}) \right) \]

(4)
HCC

The hydrostatic consistency condition (HCC) says that the relative slope of the coordinates should not be larger than unity:

\[
|\partial_x z_k| \frac{\Delta x}{\frac{1}{2}(h_{i,j,k} + h_{i+1,j,k})} \leq 1. \tag{5}
\]

It is not always possible to avoid violation of HCC. The problem may be relaxed by smoothing bathymetry, having coarse vertical near-bed resolution, applying adaptive grids, increasing horizontal resolution …