Statistical turbulence modelling in the oceanic mixed layer

Hans Burchard

hans.burchard@io-warnemuende.de

Baltic Sea Research Institute Warnemünde, Germany
Program of presentation

- NSE → Two-equation closure models
- Two-equation closure models
- General Ocean Turbulence Model (GOTM)
- Examples (observations versus simulations)
- General Estuarine Transport Model (GETM)
- Three-dimensional model study
- Conclusions
Basic approaches

Two approaches to averaged turbulence modelling:

- **Statistical turbulence modelling:** Convert NSE to Friedmann-Keller series, cut-off where suitable and parameterise unknown terms.
- **Empirical turbulence modelling:** Close equations on lowest order and parameterise relevant processes.
NSE → Reynolds equation

Momentum Equation:

\[
\partial_t v_i + v_j \partial_j v_i - \nu \partial_{jj} v_i + 2\varepsilon_{ijl} \Omega_j v_l = -\frac{\partial_i p}{\rho_0} - \frac{g_i}{\rho_0} \rho.
\]

Reynolds averaging:

\[v_i = \bar{v}_i + \tilde{v}_i, \quad \rho = \bar{\rho} + \tilde{\rho}, \ldots\]

Reynolds Equation:

\[
\partial_t \bar{v}_i + \bar{v}_j \partial_j \bar{v}_i - \partial_j (\nu \partial_j \bar{v}_i - \langle \tilde{v}_j \tilde{v}_i \rangle) + 2\varepsilon_{ijl} \Omega_j \bar{v}_l = -\frac{\partial_i \bar{p}}{\rho_0} - g_i \frac{\bar{\rho}}{\rho_0}.
\]
Reynolds Stress Equation

\[
\partial_t \langle \tilde{v}_i \tilde{v}_j \rangle + \partial_l \left( \tilde{v}_l \langle \tilde{v}_i \tilde{v}_j \rangle + \langle \tilde{v}_l \tilde{v}_i \tilde{v}_j \rangle - \nu \partial_l \langle \tilde{v}_i \tilde{v}_j \rangle \right) \\
= -\partial_l \tilde{v}_i \langle \tilde{v}_l \tilde{v}_j \rangle - \partial_l \tilde{v}_j \langle \tilde{v}_l \tilde{v}_i \rangle \\
= -\partial_l \tilde{v}_i \langle \tilde{v}_l \tilde{v}_j \rangle - \partial_l \tilde{v}_j \langle \tilde{v}_l \tilde{v}_i \rangle \\
= -2\Omega_l \left( \varepsilon_{ilm} \langle \tilde{v}_j \tilde{v}_m \rangle + \varepsilon_{jlm} \langle \tilde{v}_i \tilde{v}_m \rangle \right) \\
= -\frac{1}{\rho_0} \left\{ g_i \langle \tilde{v}_j \tilde{\rho} \rangle + g_j \langle \tilde{v}_i \tilde{\rho} \rangle \right\} - \frac{1}{\rho_0} \left( \langle \tilde{v}_i \partial_j \tilde{\rho} + \tilde{v}_j \partial_i \tilde{\rho} \rangle \right) \\
= -2\nu \left\langle (\partial_l \tilde{v}_j) (\partial_l \tilde{v}_i) \right\rangle.
\]
Algebraic SMCs

The following steps lead to different types of second-moment closures:

- Empirical closures of pressure-strain correlators.
- Neglect or simplification of advective and diffusive fluxes of second-moments.
- Neglect of rotational terms in the second-moment equations.
- Boundary layer assumption (neglect of horizontal gradients and non-hydrostatic effects).
- ... and many more details ...
Algebraic SMCs

Turbulent Fluxes:

\[ \langle \tilde{u}\tilde{w} \rangle = -\nu_t \partial_z \tilde{u}, \quad \langle \tilde{w}\tilde{T} \rangle = -\nu'_t \partial_z \tilde{T} \]

Eddy Viscosity / Eddy Diffusivity:

\[ \nu_t = c_\mu(\alpha_M, \alpha_N) \frac{k^2}{\varepsilon}, \quad \nu'_t = c'_\mu(\alpha_M, \alpha_N) \frac{k^2}{\varepsilon}. \]

Shear Number, Buoyancy Number:

\[ \alpha_M = \frac{k^2}{\varepsilon^2} M^2, \quad \alpha_N = \frac{k^2}{\varepsilon^2} N^2. \]
Stability Functions

Canuto et al. [2001]:
Exact TKE-Equation

\[
\partial_t k + \partial_j \left( \bar{v}_j k + \langle \bar{v}_j \frac{1}{2} \bar{v}_i^2 \rangle - \nu \partial_j k + \frac{1}{\rho_0} \langle \bar{v}_j \tilde{\rho} \rangle \right) = \underbrace{-\langle \bar{v}_j \bar{v}_i \rangle \partial_i \bar{v}_j}_{P} - \underbrace{\frac{g}{\rho_0} \langle \bar{v}_3 \tilde{\rho} \rangle}_{B} - \nu \langle (\partial_j \bar{v}_i)^2 \rangle,_{\text{ }}_{\varepsilon}
\]

This TKE equation will be modelled as it is given above, the only parameterisations needed are for the turbulent flux terms, for which usually the down-gradient approximation is used.
Dissipation equation

Exact form (e.g. Wilcox [1998]):

\[
\partial_t \varepsilon + \partial_j \left( \bar{v}_j \varepsilon + \langle \bar{v}_j \nu (\partial_j \bar{v}_i)^2 \rangle - \nu \partial_j \varepsilon + 2 \frac{\nu}{\rho_0} \langle \partial_i \bar{v}_j \partial_i \bar{p} \rangle \right) \\
= -2 \nu \partial_j \bar{v}_i \left( \langle \partial_i \bar{v}_k \partial_j \bar{v}_k \rangle + \langle \partial_k \bar{v}_i \partial_k \bar{v}_j \rangle \right) - 2 \nu \partial_j \bar{v}_i \langle \bar{v}_k \partial_j \bar{v}_i \rangle \\
\boxed{P_\varepsilon} \\
-2 \nu \frac{g}{\rho_0} \partial_j \langle \bar{v}_3 \partial_j \bar{p} \rangle - 2 \nu \left( \langle \partial_j \bar{v}_i \partial_k \bar{v}_i \partial_j \bar{v}_k \rangle + \nu \langle (\partial_i \bar{v}_k)^2 \rangle \right) \\
\boxed{B_\varepsilon} \\
\boxed{\varepsilon_\varepsilon}
\]

\( k-\varepsilon \) model (Launder and Spalding [1972]):

\[
\partial_t \varepsilon - \partial_z \left( \frac{\nu_t}{\sigma_\varepsilon} \partial_z \varepsilon \right) = \frac{\varepsilon}{k} \left( c_{\varepsilon 1} P + c_{\varepsilon 3} B - c_{\varepsilon 2} \varepsilon \right).
\]
Mellor-Yamada model

General relation between $k$, $\varepsilon$ and $L$:

\[
L = c_{\mu}^{3/4} \frac{k^{3/2}}{\varepsilon}
\]

$k-kL$ model (Mellor and Yamada [1982]):

\[
\partial_t (kL) - \partial_z (S_l \partial_z (kL)) =
\]

\[
\frac{L}{2} \left[ E_1 P + E_3 B - \left( 1 + E_2 \left( \frac{L}{L_z} \right)^2 \right) \varepsilon \right].
\]
Length scale equations (cont’d)

Other approaches are using equations for $\omega = \varepsilon/k$ ($k$-$\omega$ model), $k/\varepsilon$, $k^2/\varepsilon$, ..., so why not using the generalised approach of a $k^n \varepsilon^m$ equation?

Generic length scale equation (Umlauf and Burchard [2003]):

$$\partial_t (k^n \varepsilon^m) - \partial_z \left( \frac{\nu_t}{\sigma_{nm}} k^n \varepsilon^m \right) =$$

$$k^{n-1} \varepsilon^m \left( c_{nm1} P + c_{nm3} B - c_{nm2} \varepsilon \right)$$

This works without correction term only for $m \geq 0$. 
Total equilibrium \((k-\varepsilon)\)

\[
R_i = R^\text{st}_i = \frac{c_\mu}{c'_\mu} \cdot \frac{c_{2\varepsilon} - c_{1\varepsilon}}{c_{2\varepsilon} - c_{3\varepsilon}}.
\]

\(R^\text{st}_i \approx 0.25\): Steady-state Richardson number.
Kato-Phillips experiment

MLD for Canuto et al. version A

Empirical
Ri_st = 0.2
Ri_st = 0.3
Ri_st = 0.4
Ri_st = 0.6
Ri_st = 0.8
Total equilibrium \((k - kL)\)

\[ R_i = R_{st}^i = \frac{KM}{KH} \cdot \frac{1 - E_1}{1 - E_3}. \]

\(R_{st}^i \approx 0.25: \) Steady-state Richardson number.

GOTM is a one-dimensional numerical model developed and supported by a core team of ocean modellers. GOTM aims at simulating accurately vertical exchange processes in the marine environment where mixing is known to play a key role. GOTM is freely available under the GPL (Gnu Public License).

The interested user can download the source code, a set of test cases (Papa, November, Flex, ...) and a comprehensive report.

You are warmly invited to join the GOTM mailing list and send any comments/questions to the GOTM team or become a GOTM contributor. The GOTM developers are grateful to their sponsors.
Turbulence under waves

How to compare these observations?
How to simulate them??
Breaking waves: Simulations

![Graph showing the relationship between \((z - z_0)/H_s\) and \(\varepsilon / c_w u^3 / H_s\).](attachment:image.png)

- Terray et al. [1996]
- Drennan et al. [1996]
- Anis and Moum et al. [1995]
- Numerical, \(z_0/H_s = 2.0\)
- Numerical, \(z_0/H_s = 1.0\)
- Numerical, \(z_0/H_s = 0.5\)
- Log-Law
- No shear production

\[ /c_w u^3 / H_s = H_s (z_0 - z) / H_s \]
Free Convection

![Graphs of temperature, temperature flux, and dissipation rate](image)

**Temperature**
- KC
- RH
- CA
- CB
- LES

**Temperature Flux**
- KC
- RH
- CA
- CB
- LES

**Dissipation Rate**
- KC
- RH
- CA
- CB
- LES

Seminar at Bjerknes Centre, October 16, 2003, Bergen, Norway – p. 19/44
Geographical overview

Lago Maggiore  Northern North Sea
Liverpool Bay    OWS Papa →→→
Lago Maggiore, Italy

Observations and simulations of $T$ and $\varepsilon$ (Stips et al. [2002])
Open Ocean: OWS Papa

Heat budget

![Graph showing heat budget for OWS Papa 1961/62 with two lines representing heat from T-profiles and heat from surface fluxes over Julian Day 1961/62.](image-url)
Open Ocean: OWS Papa

Temperature Simulations: \( k-\varepsilon \) model with alg. SMC

Simulation OWS Papa 1961/62

Julian Day 1961/62

Depth [m]

T [°C]
Open Ocean: OWS Papa

Temperature profiles

T-profiles, day 120

T-profiles, day 150

T-profiles, day 180

T-profiles, day 210

T-profiles, day 240

T-profiles, day 270
Open Ocean: OWS Papa

Sea Surface Temperature

OWS Papa 1961/62

Observation
Simulation KC

Observation
Simulation CA

Julian Day 1961/62
Northern North Sea

Bathymetry and station map
Northern North Sea

Simulated temperature
Northern North Sea

Wind and Tides

Surface stress at station NNS

Bed stress at station NNS

Date in 1998
NPZD model in GOTM

Temperature
NPZD model in GOTM

Nutrients
NPZD model in GOTM

Phytoplankton
NPZD model in GOTM

Zooplankton
NPZD model in GOTM

Detritus
Liverpool Bay

Section of Temperature and Salinity

Rippeth, Fisher, Simpson [2001]
Liverpool Bay

Observed and simulated temperature and salinity

Simpson, Burchard, Fisher, Rippeth [2002]
Liverpool Bay

Observed and simulated current velocity

Simpson, Burchard, Fisher, Rippeth [2002]
Liverpool Bay

Observed and simulated dissipation rates

Simpson, Burchard, Fisher, Rippeth [2002]
GETM

General Estuarine Transport Model
(Burchard & Bolding [2002])

- General vertical coordinates
- Horizontal curvilinear coordinates
- Uses GOTM as turbulence model
- Drying & flooding
- Public Domain
North Sea simulation

Sea surface temperature on Aug. 1, 1997

Sea surface salinity on Aug. 1, 1997

Temperature section on Aug. 1, 1997

Salinity section on Aug. 1, 1997
North Sea / Baltic Sea

SSS on June 1, 1997, $\Delta x = \Delta y = 3$ nm

$S / \text{psu}$
Conclusions

- Reynolds decomposition provides a physically sound framework for developing turbulence models.
- Empirical parameters allow for calibrating such models to the real world.
- With such models, turbulence is described but not better understood.
- For mixed layers, two-equation models with algebraic SMCs are sufficient for reproducing observations.
- These models are economic enough for using them in 3D models for oceanic applications.