Recent advances in oceanic mixed-layer modelling

Hans Burchard\textsuperscript{1,2} and Karsten Bolding\textsuperscript{2}

hans.burchard@io-warnemuende.de, karsten@bolding-burchard.com

1. Baltic Sea Research Institute Warnemünde, Germany
2. Bolding & Burchard Hydrodynamics, Denmark/Germany
Program of presentation

- NSE $\rightarrow$ Two-equation closure models
- Two-equation closure models
- Langmuir Circulation parameterisations
- General Ocean Turbulence Model (GOTM)
- Examples (observations versus simulations)
- General Estuarine Transport Model (GETM)
- Three-dimensional model study
- Conclusions
Basic approaches

Two approaches to averaged turbulence modelling:

- **Statistical turbulence modelling:** Convert NSE to Friedmann-Keller series, cut-off where suitable and parameterise unknown terms.
- **Empirical turbulence modelling:** Close equations on lowest order and parameterise relevant processes.
NSE → Reynolds equation

Momentum Equation:

\[
\partial_t v_i + v_j \partial_j v_i - \nu \partial_{jj} v_i + 2\varepsilon_{ijl} \Omega_j v_l = -\frac{\partial_i p}{\rho_0} - \frac{g_i}{\rho_0} \rho.
\]

Reynolds averaging:

\[v_i = \bar{v}_i + \tilde{v}_i, \quad \rho = \bar{\rho} + \tilde{\rho}, \ldots\]

Reynolds Equation:

\[
\partial_t \bar{v}_i + \bar{v}_j \partial_j \bar{v}_i - \partial_j (\nu \partial_j \bar{v}_i - \langle \tilde{v}_j \tilde{v}_i \rangle) + 2\varepsilon_{ijl} \Omega_j \bar{v}_l = -\frac{\partial_i \bar{p}}{\rho_0} - g_i \frac{\bar{\rho}}{\rho_0}.
\]
Reynolds Stress Equation

\[
\partial_t \langle v_i v_j \rangle + \partial_l \left( \bar{v}_l \langle v_i v_j \rangle + \langle v_i \bar{v}_j \rangle - \nu \partial_l \langle \bar{v}_i \bar{v}_j \rangle \right) \\
= - \partial_l \bar{v}_i \langle \bar{v}_l v_j \rangle - \partial_l \bar{v}_j \langle \bar{v}_l v_i \rangle \\
\quad - 2\Omega_l (\varepsilon_{ilm} \langle \bar{v}_j \bar{v}_m \rangle + \varepsilon_{jlm} \langle \bar{v}_i \bar{v}_m \rangle) \\
\quad - \frac{1}{\rho_0} \{g_i \langle \bar{v}_j \bar{\rho} \rangle + g_j \langle \bar{v}_i \bar{\rho} \rangle \} - \frac{1}{\rho_0} (\langle \bar{v}_i \partial_j \bar{\rho} + \bar{v}_j \partial_i \bar{\rho} \rangle) \\
\quad - 2\nu \langle (\partial_l \bar{v}_j)(\partial_l \bar{v}_i) \rangle.
\]
Algebraic SMCs

The following steps lead to different types of second-moment closures:

- Empirical closures of pressure-strain correlators.
- Neglect or simplification of advective and diffusive fluxes of second-moments.
- Neglect of rotational terms in the second-moment equations.
- Boundary layer assumption (neglect of horizontal gradients and non-hydrostatic effects).
- ... and many more details ...
Algebraic SMCs

Turbulent Fluxes:

\[ \langle \tilde{u}\tilde{w} \rangle = -\nu_t \partial_z \tilde{u}, \quad \langle \tilde{w}\tilde{T} \rangle = -\nu'_t \partial_z \tilde{T} \]

Eddy Viscosity / Eddy Diffusivity:

\[ \nu_t = c_\mu(\alpha_M, \alpha_N) \frac{k^2}{\varepsilon}, \quad \nu'_t = c'_\mu(\alpha_M, \alpha_N) \frac{k^2}{\varepsilon}. \]

Shear Number, Buoyancy Number:

\[ \alpha_M = \frac{k^2}{\varepsilon^2} M^2, \quad \alpha_N = \frac{k^2}{\varepsilon^2} N^2. \]
Stability Functions

Canuto et al. [2001]:

\[ c_\mu \]

\[ c'_\mu \]

\[ \alpha_N \]

\[ \alpha_M \]
Exact TKE-Equation

\[
\partial_t k + \partial_j \left( \bar{v}_j k + \langle \bar{v}_j \frac{1}{2} \bar{v}_i^2 \rangle - \nu \partial_j k + \frac{1}{\rho_0} \langle \bar{v}_j \bar{p} \rangle \right)
\]

\[
= -\langle \bar{v}_j \bar{v}_i \rangle \partial_i \bar{v}_j - \frac{g}{\rho_0} \langle \bar{v}_3 \bar{p} \rangle - \nu \langle (\partial_j \bar{v}_i) \rangle^2 ,
\]

This TKE equation will be modelled as it is given above, the only parameterisations needed are for the turbulent flux terms, for which usually the down-gradient approximation is used.
Dissipation equation

Exact form (e.g. Wilcox [1998]):

\[
\partial_t \varepsilon + \partial_j \left( \tilde{v}_j \varepsilon + \langle \tilde{v}_j \nu (\partial_j \tilde{v}_i) \rangle - \nu \partial_j \varepsilon + 2 \frac{\nu}{\rho_0} \langle \partial_i \tilde{v}_j \partial_i \tilde{p} \rangle \right) \\
= -2\nu \partial_j \tilde{v}_i \left( \langle \partial_i \tilde{v}_k \partial_j \tilde{v}_k \rangle + \langle \partial_k \tilde{v}_i \partial_k \tilde{v}_j \rangle \right) - 2\nu \partial_j \tilde{v}_i \langle \tilde{v}_k \partial_j \tilde{v}_i \rangle \]

\[
= \underbrace{-2\nu \partial_j \tilde{v}_i \langle \tilde{v}_3 \partial_j \tilde{\rho} \rangle - 2\nu (\langle \partial_j \tilde{v}_i \partial_k \tilde{v}_i \partial_j \tilde{v}_k \rangle + \nu \langle (\partial_i \tilde{v}_k)^2 \rangle)}_{P_\varepsilon} \\
\underbrace{-2\nu \frac{g}{\rho_0} \partial_j \langle \tilde{v}_3 \partial_j \tilde{\rho} \rangle}_{B_\varepsilon} \underbrace{-2\nu (\langle \partial_j \tilde{v}_i \partial_k \tilde{v}_i \partial_j \tilde{v}_k \rangle + \nu \langle (\partial_i \tilde{v}_k)^2 \rangle)}_{\varepsilon_\varepsilon}
\]

\( k\)-\( \varepsilon \) model (Launder and Spalding [1972]):

\[
\partial_t \varepsilon - \partial_z \left( \frac{\nu_t}{\sigma_\varepsilon} \partial_z \varepsilon \right) = \frac{\varepsilon}{k} \left( c_{\varepsilon 1} P + c_{\varepsilon 3} B - c_{\varepsilon 2} \varepsilon \right).
\]
Mellor-Yamada model

General relation between $k$, $\varepsilon$ and $L$:

$$L = c_{\mu}^{3/4} \frac{k^{3/2}}{\varepsilon}$$

$k$-$kL$ model (Mellor and Yamada [1982]):

$$\partial_t (kL) - \partial_z (S_l \partial_z (kL)) =$$

$$\frac{L}{2} \left[ E_1 P + E_3 B - \left( 1 + E_2 \left( \frac{L}{L_z} \right)^2 \right) \varepsilon \right].$$
Length scale equations (cont’d)

Other approaches are using equations for $\omega = \varepsilon / k$ ($k$-$\omega$ model), $k / \varepsilon$, $k^2 / \varepsilon$, ..., so why not using the generalised approach of a $k^n \varepsilon^m$ equation?

Generic length scale equation
(Umlauf and Burchard [2003]):

$$\partial_t (k^n \varepsilon^m) - \partial_z \left( \frac{\nu_t}{\sigma_{nm}} k^n \varepsilon^m \right) =$$

$$k^{n-1} \varepsilon^m \left( c_{nm1} P + c_{nm3} B - c_{nm2} \varepsilon \right)$$

This works without correction term only for $m \geq 0$. 
Total equilibrium ($k$-$\varepsilon$)

\[
\dot{k} = \dot{\varepsilon} \implies R_i = R_{i}^{st} = \frac{C_{\mu}}{C_{\mu}'} \cdot \frac{C_{2\varepsilon} - C_{1\varepsilon}}{C_{2\varepsilon} - C_{3\varepsilon}}.
\]

$R_{i}^{st} \approx 0.25$: Steady-state Richardson number.
Kato-Phillips experiment

MLD for Canuto et al. version A

MLD / m

Time / h

Empirical
Ri_st = 0.2
Ri_st = 0.3
Ri_st = 0.4
Ri_st = 0.6
Ri_st = 0.8
Langmuir Circulation

Stokes drift velocity:

\[
\{U_s, V_s\} = \left\{ \left( \frac{k_x h_s}{2} \right)^2, \left( \frac{k_y h_s}{2} \right)^2 \right\} \sqrt{\frac{g}{k}} \exp(-2kz) \tag{1}
\]

Modified Reynolds equations:

\[
\partial_t \bar{v}_i + \bar{v}_j \partial_j \bar{v}_i - \partial_j (\nu \partial_j \bar{v}_i - \langle \bar{v}_j \bar{v}_i \rangle) + 2\varepsilon_{ijl} \Omega_j \bar{v}_l
\]

\[
= -\frac{\partial_i \bar{p}}{\rho_0} - g_i \frac{\bar{\rho}}{\rho_0} + \varepsilon_{ijl} \{(U_s)_j \varepsilon_{lmn} \partial_m \bar{u}_n\}.
\]

Vortex force term:

\[
\varepsilon_{ijl} \{ (U_s)_j \varepsilon_{lmn} \partial_m \bar{u}_n \} = \vec{U}_s \times (\vec{\nabla} \times \vec{u}). \tag{2}
\]
Langmuir Circulation, cont’d

Langmuir Circulation (LC) in TKE models

TKE-equation (*Kanthan and Clayson*, [2003]):

\[
\frac{\partial_t k}{\partial z} = \left(\frac{\nu_t}{\sigma_k} \frac{\partial_z k}{\partial z}\right) \\
= \nu_t \left( (\partial_z \bar{u})^2 + (\partial_z \bar{v})^2 \right) + 2\nu_t (U_s \partial_z \bar{u} + V_s \partial_z \bar{v}) \\
+ \frac{g}{\rho_0} \nu' \rho \partial_z \bar{\rho} - \varepsilon.
\]
Empirical model: KPP

K-Profile Parameterisation Model

\[ \langle \tilde{w}\tilde{x} \rangle = -K_x \partial_z \tilde{x} + K_x \gamma_x, \quad \text{local} \quad \text{non-local} \] (3)

\[ K_x = h(t) w_x(z,t) G_x(z,t) \] (4)

\[ h(t) \quad \text{mixed-layer depth} \]

\[ w_x(z,t) = \frac{\kappa u_*(t)}{\phi_x(z,t)} \quad \text{turbulent velocity scale} \]

\[ G_x(z,t) \quad \text{shape function} \]

\[ \partial_z \tilde{x} = -\frac{\langle \tilde{w}\tilde{x} \rangle_0 \phi_x(z/L)}{u_* \kappa |z|} \quad \text{structure function } \phi_x \] (5)
KPP continued

**Langmuir Circulation (LC) in KPP**

Enhanced mixing due to LC by *Smyth et al.* [2002]:

\[
wx(z, t) = \frac{\kappa u_*(t)}{\phi_x(z, t)} \left\{ 1 + \frac{C_w(u_*, w_*)}{La^4} \right\}^{1/2}
\]

(6)

with

\[
C_w(u_*, w_*) = C_{w0} \left[ \frac{u_*^3}{u_*^3 + 0.6w_*^3} \right]^l,
\]

(7)

Deardorff velocity scale:

\[
w_* = (\kappa B_0 h)^{1/3},
\]

Langmuir number:

\[
La = \left( \frac{u_*}{|\bar{U}_s(0)|} \right)^{1/2}
\]

(8)
GOTM is a one-dimensional numerical model developed and supported by a core team of ocean modellers. GOTM aims at simulating accurately vertical exchange processes in the marine environment where mixing is known to play a key role. GOTM is freely available under the GPL (Gnu Public License).

The interested user can download the source code, a set of test cases (Papa, November, Flex, ...) and a comprehensive report.

You are warmly invited to join the GOTM mailing list and send any comments/questions to the GOTM team or become a GOTM contributor. The GOTM developers are grateful to their sponsors.
Free Convection

Temperature

\[ \frac{z}{D_m} \]

Temperature Flux

\[ \frac{z}{D_m} \]

Dissipation Rate

\[ \frac{\varepsilon}{B_0} \]
Lago Maggiore, Italy

Observations and simulations of $T$ and $\varepsilon$ (Stips et al. [2002])
Open Ocean: OWS Papa

Temperature Observations

Measurements OWS Papa 1961/62

T [°C]

Depth [m]

Julian Day 1961/62
Open Ocean: OWS Papa

Heat budget

![Diagram showing heat budget for OWS Papa 1961/62](image)
Open Ocean: OWS Papa

Temperature Simulations: $k$-$\varepsilon$ model with alg. SMC

Simulation OWS Papa 1961/62

T [°C]

Depth [m]

Julian Day 1961/62
Open Ocean: OWS Papa

Temperature profiles
Open Ocean: OWS Papa

Sea Surface Temperature

OWS Papa 1961/62

Observation
Simulation KC

Observation
Simulation CA

Julian Day 1961/62
GETM

General Estuarine Transport Model
(Burchard & Bolding [2002])

- General vertical coordinates
- Horizontal curvilinear coordinates
- Uses GOTM as turbulence model
- Drying & flooding
- Using MPI for parallel runs
- Public Domain
Conclusions

- Reynolds decomposition provides a physically sound framework for developing turbulence models.
- Empirical parameters allow for calibrating such models to the real world.
- Langmuir Circulation parameterisations in KPP and TKE models are not mature yet.
- For mixed layers, two-equation models with algebraic SMCs are sufficient for reproducing observations.
- These models are economic enough for using them in 3D models for oceanic applications.