Two-equation turbulence models for boundary layer flows

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Program of presentation
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- NSE $\rightarrow$ Two-equation models
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- Boundary conditions
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- Modelling of Estuarine Turbidity Maxima
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- Examples (observations versus simulations)
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- Modelling of Estuarine Turbidity Maxima
- Conclusions
Navier-Stokes-Equations

Continuity Equation:

\[ \partial_j v_j = 0 \]
Navier-Stokes-Equations

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$$\partial_j v_j = 0$$

Momentum Equation:

$$\partial_t v_i + v_j \partial_j v_i - \nu \partial_{jj} v_i + 2\varepsilon_{ijl} \Omega_j v_l = -\frac{\partial_i p}{\rho_0} - \frac{g_i}{\rho_0}$$
Navier-Stokes-Equations

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Heat Equation:

\[ \partial_t T + v_j \partial_j T - \nu' \partial_{jj} T = \frac{\partial_z I}{c'_p \rho_0}, \]
Averaging Rules

1. Linearity:

\[ \langle U + \lambda V \rangle = \langle U \rangle + \lambda \langle V \rangle. \]
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4. Products of Averages:
\[ \langle U \langle V \rangle \rangle = \langle U \rangle \langle V \rangle \]
Reynolds-averaged Equations

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\[ \partial_j \bar{v}_j = 0 \]
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Reynolds Equation:

\[ \partial_t \bar{v}_i + \bar{v}_j \partial_j \bar{v}_i - \partial_j (\nu \partial_j \bar{v}_i - \langle \bar{v}_j \bar{v}_i \rangle) + 2\varepsilon_{i,jl} \Omega_j \bar{v}_l = -\frac{\partial_i \bar{p}}{\rho_0} - g_i \frac{\bar{\rho}}{\rho_0} \]
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Heat Equation:

\[ \partial_t \tilde{T} + v_j \partial_j \tilde{T} - \partial_j(\nu' \partial_j \tilde{T} - \langle \tilde{v}_j \tilde{T} \rangle) = \frac{\partial_z I}{c'_p \rho_0} \]
Reynolds Stress Equation

\[ \partial_t \langle \tilde{v}_i \tilde{v}_j \rangle + \partial_l \left( \tilde{v}_i \langle \tilde{v}_i \tilde{v}_j \rangle + \langle \tilde{v}_i \tilde{v}_i \tilde{v}_j \rangle - \nu \partial_l \langle \tilde{v}_i \tilde{v}_j \rangle \right) = -\partial_l \tilde{v}_i \langle \tilde{v}_l \tilde{v}_j \rangle - \partial_l \tilde{v}_j \langle \tilde{v}_l \tilde{v}_i \rangle \]

\[ P_{ij} \]

\[ -2\Omega_l \left( \varepsilon_{ilm} \langle \tilde{v}_j \tilde{v}_m \rangle + \varepsilon_{jlm} \langle \tilde{v}_i \tilde{v}_m \rangle \right) \]

\[ \Omega_{ij} \]

\[ \frac{1}{\rho_0} \left\{ g_i \langle \tilde{v}_j \tilde{\rho} \rangle + g_j \langle \tilde{v}_i \tilde{\rho} \rangle \right\} - \frac{1}{\rho_0} \left( \langle \tilde{v}_i \partial_j \tilde{\rho} + \tilde{v}_j \partial_i \tilde{\rho} \rangle \right) \]

\[ B_{ij} \]

\[ -2\nu \left\langle \left( \partial_l \tilde{v}_j \right) \left( \partial_l \tilde{v}_i \right) \right\rangle . \]

\[ \varepsilon_{ij} \]

\[ \Pi_{ij} \]
Heat Flux Equation

\[
\begin{align*}
\partial_t \langle \tilde{v}_i \tilde{T} \rangle + \partial_j \left( \tilde{v}_j \langle \tilde{v}_i \tilde{T} \rangle + \langle \tilde{v}_i \tilde{v}_j \tilde{T} \rangle - (\nu + \nu') \partial_j \langle \tilde{v}_i \tilde{T} \rangle \right) \\
+ \nu' \langle \tilde{T} \partial_{jj} \tilde{v}_i \rangle + \nu \langle \tilde{v}_i \partial_{jj} \tilde{T} \rangle \\
= - \langle \tilde{v}_i \tilde{v}_j \rangle \partial_j \tilde{T} - \langle \tilde{v}_j \tilde{T} \rangle \partial_j \tilde{v}_i - 2\varepsilon_{ijl} \Omega_j \langle \tilde{v}_i \tilde{T} \rangle \\
- \frac{g_i}{\rho_0} \langle \tilde{T} \tilde{\rho} \rangle - \frac{1}{\rho_0} \langle \tilde{T} \partial_i \tilde{p} \rangle - 2(\nu + \nu') \langle (\partial_j \tilde{v}_i) (\partial_j \tilde{T}) \rangle
\end{align*}
\]
Algebraic SMCs

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- Assumption of local equilibrium for tracer variances.
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- Neglect of rotational terms in the second-moment equations.
- Neglect of tracer-tracer correlations.
- Assumption of local equilibrium for tracer variances.
- Boundary layer assumption (neglect of horizontal gradients and non-hydrostatic effects).
Algebraic SMCs

Turbulent Fluxes:

\[
\langle \tilde{u}\tilde{w} \rangle = -\nu_t \partial_z \tilde{u}, \quad \langle \tilde{w}\tilde{T} \rangle = -\nu'_t \partial_z \tilde{T}
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Algebraic SMCs

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Eddy Viscosity / Eddy Diffusivity:

\[
\nu_t = c_\mu(\alpha_M, \alpha_N) \frac{k^2}{\varepsilon}, \quad \nu'_t = c'_\mu(\alpha_M, \alpha_N) \frac{k^2}{\varepsilon}.
\]
Algebraic SMCs

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\]

Shear Number, Buoyancy Number:

\[
\alpha_M = \frac{k^2}{\varepsilon^2} M^2, \quad \alpha_N = \frac{k^2}{\varepsilon^2} N^2.
\]
Stability Functions

Kantha & Clayson [1994]:

\[ c_\mu \]

\[ c'_\mu \]

\[ \alpha_N \]

\[ \alpha_M \]
Stability Functions

Canuto et al. [2001]:

\[ c_\mu \]

\[ c'_\mu \]

\[ \alpha_M \]

\[ \alpha_N \]
Stability functions (qe)

Kantha & Clayson [1994]

Canuto et al. [2000], model A
This TKE equation will be modelled as given above, the only parameterisations needed are for the turbulent flux terms, for which usually the down-gradient approximation is used.
Exact TKE-Equation

\[ \partial_t k + \partial_j \left( \bar{\nu}_j k + \langle \bar{\nu}_j \frac{1}{2} \bar{\nu}_i^2 \rangle - \nu \partial_j k + \frac{1}{\rho_0} \langle \bar{\nu}_j \bar{p} \rangle \right) \]

\[ = -\langle \bar{\nu}_j \bar{\nu}_i \rangle \partial_i \bar{\nu}_j - \frac{g}{\rho_0} \langle \bar{\nu}_3 \bar{p} \rangle - \nu \langle (\partial_j \bar{\nu}_i)^2 \rangle, \]

This TKE equation will be modelled as it is given above, the only parameterisations needed are for the turbulent flux terms, for which usually the down-gradient approximation is used.
Length scale equations

$k$-$\varepsilon$ model (Launber and Spalding [1972]):

$$
\partial_t \varepsilon - \partial_z \left( \frac{\nu_t}{\sigma_\varepsilon} \partial_z \varepsilon \right) = \frac{\varepsilon}{k} \left( c_{\varepsilon 1} P + c_{\varepsilon 3} B - c_{\varepsilon 2} \varepsilon \right).
$$
Length scale equations

\( k-\varepsilon \) model (Lauder and Spalding [1972]):

\[
\partial_t \varepsilon - \partial_z \left( \frac{\nu_t}{\sigma_\varepsilon} \partial_z \varepsilon \right) = \frac{\varepsilon}{\kappa} (c_{\varepsilon 1} P + c_{\varepsilon 3} B - c_{\varepsilon 2} \varepsilon).
\]

\( k-kL \) model (Mellor and Yamada [1982]):

\[
\partial_t (kL) - \partial_z (S_i \partial_z (kL)) = \\
\frac{L}{2} \left[ E_1 P + E_3 B - \left( 1 + E_2 \left( \frac{L}{L_z} \right)^2 \right) \varepsilon \right].
\]
Total equilibrium ($k$-$\varepsilon$)

\[ R_i = R_i^{st} = \frac{c_\mu}{c'_\mu} \cdot \frac{c_{2\varepsilon} - c_{1\varepsilon}}{c_{2\varepsilon} - c_{3\varepsilon}}. \]

\[ R_i^{st} \approx 0.25: \text{Steady-state Richardson number.} \]
Total equilibrium ($k$-$\varepsilon$)

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$R_i^{st} \approx 0.25$: Steady-state Richardson number.
Total equilibrium ($k - kL$)

\[ R_i = R_{i}^{st} = \frac{K_M}{K_H} \cdot \frac{1 - E_1}{1 - E_3}. \]

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Total equilibrium \((k-kL)\)

\[ R_i = R_{i}^{st} = \frac{K_M}{K_H} \cdot \frac{1 - E_1}{1 - E_3}. \]

\( R_{i}^{st} \approx 0.25: \) Steady-state Richardson number.
Length scale equations (cont’d)

Generic length scale equation (Umlauf and Burchard [2002]):

\[ \partial_t (k^n \varepsilon^m) - \partial_z \left( \frac{\nu_t}{\sigma_{nm}} k^n \varepsilon^m \right) = \]

\[ k_i^{n-1} \varepsilon^m (c_{nm1} P + c_{nm3} B - c_{nm2} \varepsilon) \]
Length scale equations (cont’d)

Generic length scale equation (*Umlauf and Burchard* [2002]):

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\partial_t (k^n \varepsilon^m) - \partial_z \left( \frac{\nu_t}{\sigma_{nm}} k^n \varepsilon^m \right) =
\]

\[
k_{\varepsilon}^{n-1} \varepsilon^m (c_{nm1} P + c_{nm3} B - c_{nm2} \varepsilon)
\]

General relation between \( k, \varepsilon \) and \( L \):

\[
L = c^{3/4} \frac{k^{3/2}}{\varepsilon}
\]
Boundary conditions

Law of the wall:

\[ k(z') = \frac{u_*^2}{C_{\mu}^{1/2}}; \quad \varepsilon(z') = \frac{u_*^3}{\kappa(z' + z_0)}; \quad l(z') = \kappa(z' + z_0) \]
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Breaking surface waves:

\[ k(z') = K(z' + z_0)^\alpha; \quad \varepsilon(z') = E(z' + z_0)^\beta; \quad l(z') = L(z' + z_0) \]
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\[-2.5 < \alpha - 2.0; \quad -5.0 < \beta < -2.0; \quad L \approx 0.2\]
GOTM is a one-dimensional numerical model developed and supported by a core team of ocean modellers. GOTM aims at simulating accurately vertical exchange processes in the marine environment where mixing is known to play a key role. GOTM is freely available under the GPL (Gnu Public License).

The interested user can download the source code, a set of test cases (Papa, November, Flex, ...) and a comprehensive report.

You are warmly invited to join the GOTM mailing list and send any comments/questions to the GOTM team or become a GOTM contributor. The GOTM developers are grateful to their sponsors.
Wave-enhanced layer

Simulation with the generic two-equation model by Umlauf and Burchard [2002]:

\[
\frac{z_0 - z}{H_s} = \frac{\varepsilon}{(c_w u_*^3/H_s)}
\]

- Terray et al. [1996]
- Drennan et al. [1996]
- Anis and Moum [1995]
- Numerical, \( z_0/H_s = 2.0 \)
- Numerical, \( z_0/H_s = 1.0 \)
- Numerical, \( z_0/H_s = 0.5 \)
- Log-Law
- No shear production
Free Convection

Temperature

Temperature Flux

Variance of $u$

Variance of $w$

Variance of $T$

Dissipation Rate

$(\bar{T} - T_{\text{max}})/T_*$

$\langle \bar{w} \bar{T} \rangle / (w_* T_*)$

$\langle \bar{u} \rangle / (u_* T_*)$

$\langle w^2 \rangle / (w_*^2)$

$\langle T^2 \rangle / (T_*^2)$

$\varepsilon / B_0$
Lago Maggiore, Italy

Observations and simulations of $T$ and $\varepsilon$ (Stips et al. [2002])
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Northern North Sea

Bathymetry and station map
Northern North Sea

Bathymetry and station map
Northern North Sea

Wind and Tides

Surface stress at station NNS

Bed stress at station NNS

Date in 1998

Northern North Sea

Dissipation Rate Observed from FLY, log10(W/kg)

Dissipation Rate Observed from MST, log10(W/kg)

Dissipation Rate Simulated by CA, log10(W/kg)

Dissipation Rate Simulated by AG, log10(W/kg)
Liverpool Bay

Section of Temperature and Salinity

Fig 2.a  Temperature (Degrees C)

Fig 2.b  Salinity (PSU)

Rippeth, Fisher, Simpson [2001]
Liverpool Bay

Observed and simulated temperature and salinity

Simpson, Burchard, Fisher, Rippeth [2002]
Liverpool Bay

Observed and simulated current velocity

Simpson, Burchard, Fisher, Rippeth [2002]
Liverpool Bay

Observed and simulated dissipation rates

Simpson, Burchard, Fisher, Rippeth [2002]
Liverpool Bay

Observed and simulated dissipation rates

Simpson, Burchard, Fisher, Rippeth [2002]
Links to Sediment Modelling

- Lagrangian or Eulerian modelling?
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- How to model the wave-BBL?
Conceptual model for ETMs

Jay & Musiak, 1994
ETM computer simulations

Burchard & Baumert, 1998

Slack tide flood

Full ebb

Slack tide ebb

Full flood

Ruiz Villareal and Burchard, under progress
Conclusions

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• Two-equation turbulence models provide a useful tool for investigating and reproducing various processes in boundary layer flows.
Conclusions

- With statistical turbulence modelling, we do not learn about the structure of turbulence but about the impact of turbulence on the flow.
- The choices for the length scale equation and the algebraic second-moment closure are independent.
- Two-equation turbulence models provide a useful tool for investigating and reproducing various processes in boundary layer flows.
- Estuarine Turbidity Maxima have been investigated in some detail with these models.