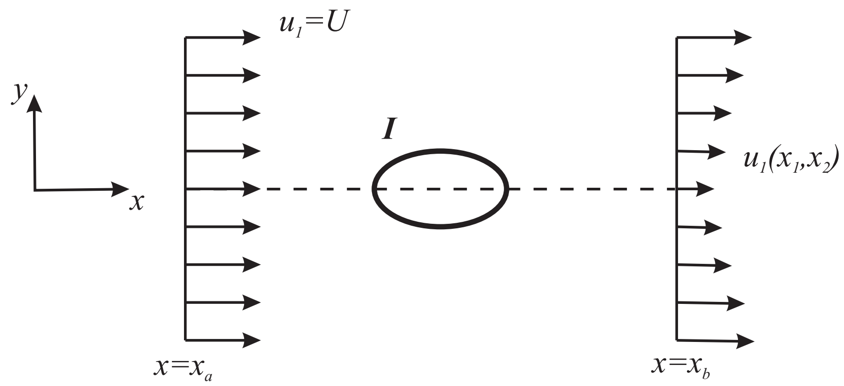
Assignment Nr. 4

due December 06th

Problem 1



Consider a cylindrical island, I, located in a stationary velocity field, $\boldsymbol{u}(\boldsymbol{x})$, with constant density, $\rho = \rho_0$. For simplicity, it is assumed that the island is symmetric about the y-axis and has no dependence on the z-direction (pointing out of the page). At $x = x_a$, upstream of the island, the velocity field is given by $\boldsymbol{u}(\boldsymbol{x}) = U\boldsymbol{e}_1$, where the reference velocity, U, is constant. At $x = x_b$, far enough downstream of the island, it can be assumed with good accuracy that the velocity is again parallel to the x-axis, $\boldsymbol{u}(\boldsymbol{x}) = u_1(x_b, y)\boldsymbol{e}_1$, where $u_1(x_b, y) \leq U$.

(a) Design a control volume bounded by the surface of the island, A_I, the surfaces at $x = x_a$ and $x = x_b$, the surfaces at $y = \infty$ and $y = -\infty$, and an infinitely narrow slit connecting the surface at $y = \infty$ and the surface of the island, A_I. (Obviously, the surfaces at $y = \pm\infty$ should be drawn symbolically at some value $y = \pm h$.)

(b) Show that, for this problem, the integral version of the balance of mass reduces to

$$\int_A \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}A = 0 \qquad (1)$$

where A denotes the total surface of the control volume, and $\boldsymbol{n}$ the local outward unit normal vector on that surface.

(c) Show that evaluating (1) on all surfaces of the control volume yields

$$\int_{-\infty}^{\infty} [U - u_1(x_b, y)] \, \mathrm{d}y = 2 \int_{x_a}^{x_b} u_2(x, \infty) \mathrm{d}x \; , \tag{2}$$

where the symmetry of velocity distribution about the y-axis has been exploited. What is the physical meaning of this result?

Problem 2

Consider again the current arround the cylindrical island discussed in problem 2. The same simplifications (stationarity, constant density, symmetry about y-axis) apply, and all physical quantities are identical to those in problem 2. The goal is now to derive an expression for the *drag force*, $\boldsymbol{F}$, exerted by the fluid on the island.

(a) Start by showing that, for this problem (assuming that body forces are negligible), the integral balance of momentum can be written as

$$\int_A \rho_0 \boldsymbol{u}(\boldsymbol{u} \cdot \boldsymbol{n}) \, \mathrm{d}A = \int_A \boldsymbol{t} \, \mathrm{d}A \; , \tag{3}$$

where $\boldsymbol{t}$ is the stress vector, and A the total surface of the control volume, which is identical to that designed in problem 2a.

(b) Assume that at the surfaces $x = x_a$, $x = x_b$, and $y = \pm\infty$, the stress vector is of the form $\boldsymbol{t} = -p_0 \boldsymbol{n}$, where p_0 is the constant pressure far away from the island. Show that the contributions of all surfaces to the integral on the right hand side of (3) cancel, except for the term

$$\int_{A_I} \boldsymbol{t} \, \mathrm{d}A \; , \tag{4}$$

where A_I denotes the surface of the island. Argue that this integral is just the negative of the drag force, $\boldsymbol{F}$, exerted by the fluid on the island. Thus, the right hand side of (3) can be written as $-\boldsymbol{F}$.

(c) Evaluate the left hand side of (3) on all surfaces of the control volume. Take the scalar product of the result with $\boldsymbol{e}_1$ to show that the x-component of the drag force, $\boldsymbol{F}$, is given by

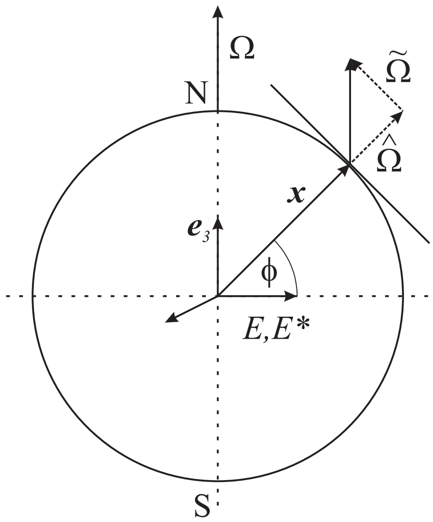
$$F_1 = \rho_0 U^2 \int_{-\infty}^{\infty} \left(1 - \frac{u_1(x_b, y)^2}{U^2} \right) \mathrm{d}y - 2\rho_0 U \int_{x_a}^{x_b} u_2(x, \infty) \, \mathrm{d}x \; , \tag{5}$$

where again the symmetry of the velocity field about the x-axis has been exploited. Hint: use the fact the at $y = \pm\infty$, the velocity is of the form $\boldsymbol{u} = U\boldsymbol{e}_1 + u_2(x, \pm\infty)\boldsymbol{e}_2$.

(d) Use the balance of mass, (2), to show that (5) can be re-expressed as

$$F_1 = \rho_0 U^2 \int_{-\infty}^{\infty} \frac{u_1(x_b, y)}{U} \left(1 - \frac{u_1(x_b, y)}{U}\right) \mathrm{d}y \quad . \tag{6}$$

Problem 3



Assume that the origins of a fixed reference system, E (with fixed base vectors, $\boldsymbol{e}_i$), and a rotating reference system, E^* (with rotating base vectors, $\boldsymbol{e}_i^*$), are located at the center of a rotating spherical planet. At the time considered, the base vectors of both coordinate systems coincide, and the base vectors $\boldsymbol{e}_3$ and $\boldsymbol{e}_3^*$ point into the direction of the planet's rotation axis indicated by $\boldsymbol{\Omega}$. The reference system E^* rotates with the planet around the same axis with constant angular velocity $|\boldsymbol{\Omega}| = 2\pi/T$, where T is the time for one complete revolution of the planet.

Let $\boldsymbol{x}$ denote the position vector (with respect to the rotating reference system) of a point located at the planet's surface at the latitude ϕ. $\boldsymbol{\Omega}$ can be split into two components, one inside and one perpendicular to the planet's tangent plane at $\boldsymbol{x}$, such that $\boldsymbol{\Omega} = \tilde{\boldsymbol{\Omega}} + \hat{\boldsymbol{\Omega}}$.

(a) Argue that for the geometry discussed above, the only accelerations resulting from the motion of the reference system are the *centrifugal acceleration*, $-\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x})$, and the *Coriolis acceleration*, $-2\boldsymbol{\Omega} \times \boldsymbol{u}$. Explicitly discuss for which reason(s) the other system acceleration terms vanish.

(b) Let's first consider only the Coriolis acceleration due to $\hat{\boldsymbol{\Omega}}$, perpendicular to the tangent plane. Which of the following statements are correct for a point with position vector $\boldsymbol{x}$?

(i) A point moving in the tangent plane always feels a Coriolis acceleration towards the center of the planet, except at the poles.

(ii) A point moving in the tangent plane always feels a Coriolis acceleration towards the right on the northern hemisphere and towards the left on the southern hemisphere. At the equator, it feels no Coriolis acceleration at all.

(iii) A point moving towards the center of the planet always feels a Coriolis acceleration towards the center of the planet, except at the equator.

(c) Next, consider only the Coriolis acceleration due to $\tilde{\mathbf{\Omega}}$, pointing towards north. Which of the following statements are correct?

(i) A point moving in the tangent plane always feels a Coriolis acceleration away from the center, except at the poles, or if it moves exactly towards the North.

(ii) A point moving towards the center of the planet always feels a Coriolis acceleration towards the East (i.e. into the paper), except at the poles.

(iii) At the equator, a point moving towards the East always feels a Coriolis acceleration towards the North.

(d) In what directions point the components of the Coriolis acceleration resulting from $\hat{\mathbf{\Omega}}$ and $\tilde{\mathbf{\Omega}}$, respectively, for a point moving on the northern hemisphere in the tangent plane towards the East, i.e. into the paper?

(e) If an air particle is moving in the atmosphere of the planet Earth at the latitude of Rostock (about 54° N) and moves on a windy day with a speed of 10 m s^{-1}, what are the magnitudes of the Coriolis accelerations computed in (d)? A typical value for the pressure gradient between a high and a low pressure area in the atmosphere is $1/\rho\, \partial p / \partial x = 10^{-3}$ m s^{-2}. How does this value compare to the Coriolis acceleration in the tangent plane?