A note on the description of mixing in stratified layers without shear in large-scale ocean models

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ABSTRACT

Large-scale geophysical flows often exhibit layers with negligible vertical shear and infinite gradient Richardson number, $Ri$. It is well-known that these layers may be regions of active mixing even in the absence of local shear production of turbulence since, among other processes, turbulence may be supplied by vertical turbulent transport from neighboring regions. This observation is contrasted by the behavior of most turbulence parameterizations used in ocean climate modeling, predicting the collapse of mixing of mass and matter if the Richardson number exceeds a critical threshold. Here, the performance of a simple model without critical Richardson number is evaluated, taking into account the diffusion of turbulence into layers without shear production, and therefore avoiding the suppression of mixing at large values of $Ri$. The model is based on the framework of second-moment turbulence closures, focusing on the consistent modeling of the turbulent length scale for strongly stratified turbulence. Results are compared to eddy-resolving simulations of stratified shear flows that have recently become available. The model is simple enough for inclusion in ocean climate models.

1. Introduction

Turbulence closure models used in ocean climate modeling often compute the turbulent diffusivity as a function of the gradient Richardson number, $Ri = N^2/S^2$, where $N$ and $S$ denote the buoyancy frequency and the total vertical shear, respectively. Except for some recent suggestions discussed below, models of this type predict the collapse of turbulent transport in stratified regions with negligible shear although there is clear observational
evidence that layers with $Ri \to \infty$ may be turbulent. This property is shared among a variety of mixing models that are based on rather different physical concepts (e.g. Pacanowski and Philander 1981; Large et al. 1994; Canuto et al. 2001). A consistent description of mixing for large values of $Ri$ is, however, of considerable practical interest since stratified mixing layers with negligible shear are frequently observed in large-scale flows relevant to ocean climate modeling, well-known examples being the velocity maximum observed in dense bottom gravity currents (overflows) and intrusions, and the core of the jet-like equatorial undercurrent.

The shear production of turbulent kinetic energy (TKE) in layers with $Ri \to \infty$ is, by definition, negligible. In such a situation, two important alternative energy sources have been identified in geophysical flows: the first is the vertical transport of TKE from neighboring regions by turbulent motions, often referred to as self-advecton of turbulence. The second, which will not be discussed here, relates to the presence of internal waves. The importance of turbulence self-advection was also emphasized in a recent numerical study by Jackson et al. (2008), JHL in the following, who investigated a turbulent stratified jet with $Ri \to \infty$ that can be considered as a prototype for many geophysical applications. Based on their results, JHL have also suggested a simple turbulence model, explicitly designed for applications in large-scale ocean modeling, that predicts non-vanishing mixing for infinite Richardson numbers as discussed in more detail below.

The critical role of the model behavior for large values of $Ri$ has also been recognized in second-moment turbulence modeling, which will be the focus of this investigation. The pioneering models of the Mellor-Yamada type (Mellor and Yamada 1974, 1982; Kantha and Clayson 1994) predict a collapse of turbulence above a critical Richardson number of
\( \text{Ri}_c \approx 0.2 \), a value that is now considered as too low to be consistent with available data. A more consistent threshold, \( \text{Ri}_c \approx 1 \), was obtained with more advanced second-moment closures (Canuto et al. 2001; Cheng et al. 2002). However, also these models are challenged by a growing body of experimental and observational data indicating that stratified turbulence may exist even for \( \text{Ri} \gg 1 \), which has motivated the development of some recent turbulence models without a finite \( \text{Ri}_c \) (Canuto et al. 2008; Zilitinkevich et al. 2007; Galperin et al. 2007). In contrast to these investigations, emphasizing the role of the return-to-isotropy time scales in the second-moment equations, it will be shown here that a simple, consistent modeling of the turbulent length scale for strongly stratified flows also yields a model without \( \text{Ri}_c \), however, without any modifications of the equations for the second moments.

In the following it will be demonstrated that a model with structure, complexity and properties comparable to that of JHL can be derived from the second-moment closure framework, avoiding many of the ad-hoc modeling assumptions of JHL. This model is tested against the eddy resolving simulations of JHL, which provide rather unique estimates for the transport terms in the TKE budget (pressure transport and triple velocity correlations) that are notoriously difficult to measure in situ with available instrumentation.

2. Modeling approaches

The starting point of virtually all geophysical turbulence models is the transport equation for the turbulent kinetic energy,

\[
\frac{Dk}{Dt} = D_k + P + G - \varepsilon , \quad (1)
\]
where $\frac{D}{Dt}$ denotes the material derivative, $D_k$ the sum of the turbulent and viscous transport terms, and $\varepsilon$ the dissipation rate. The terms $P$ and $G$ correspond to the shear and buoyancy production of TKE, which, for flows with large aspect ratio, may be modeled according to

$$ P = \nu_t S^2, \quad G = -\nu^\theta N^2, $$

with $\nu_t$ and $\nu^\theta$ denoting the vertical diffusivities of momentum and heat. For the transport term, $D_k$, models of different complexity have been suggested but the most commonly used model for stratified shear flows corresponds to a simple down-gradient formulation,

$$ D_k = \frac{\partial}{\partial z} \left( \nu_k \frac{\partial k}{\partial z} \right), $$

where $z$ is the vertical coordinate, and $\nu_k$ denotes the turbulent diffusivity of TKE. For the stably stratified flows discussed here, the applicability of this simple model will be justified below by direct comparison with the transport of TKE diagnosed from the eddy-resolving simulations of JHL. In other applications, in particular those involving unstable stratification, more complex models are required, a fact that does, however, not affect the main conclusions derived here.

Additional model assumptions are required in order to close (1), in particular a model for the dissipation rate, $\varepsilon$, and the turbulent diffusivities $\nu_t$, $\nu^\theta$, and $\nu_k$. While there is general agreement about the form of (1), numerous suggestions have been formulated for these additional closure assumptions that will be discussed in the following.
3. Second-moment models

An important class of geophysical turbulence models is based on the transport equations for the second moments of fluctuating quantities. After introducing appropriate closure assumptions, and after a number suitable simplifications (see Umlauf and Burchard 2005), these models can be expressed in the surprisingly simple form,

\[ \nu_t = c_\mu(S, N) \nu_0 , \quad \nu_\theta = c_\theta(S, N) \nu_0 , \]  \hspace{1cm} (4)

where

\[ \nu_0 = \frac{k^2}{\varepsilon} , \]  \hspace{1cm} (5)

and \( c_\mu \) and \( c_\theta \) are so-called stability functions depending on the non-dimensional shear and buoyancy numbers, \( S = Sk/\varepsilon \) and \( N = Nk/\varepsilon \). If the so-called 'quasi-equilibrium' assumption (Galperin et al. 1988) is applied, (4) may be simplified to a functional dependency on a single parameter, \( N \). This may be interpreted as a dependency on the turbulence Froude number, \( Fr_t \propto N^{-\frac{1}{2}} \), which has physical support (Ivey and Imberger 1991), and leads to a simple and attractive model as shown below. All results presented here will be based on the stability functions originally presented by Canuto et al. (2001); shape and properties of these and other types of stability functions are discussed e.g. in Burchard and Bolding (2001) and Umlauf and Burchard (2005).

The dissipation rate, \( \varepsilon \), required in (1) and (4), may be computed from different types of transport equations (see e.g. Umlauf and Burchard 2003; Umlauf et al. 2003). Here, the 'classical' form,

\[ \frac{D\varepsilon}{Dt} = D_\varepsilon + \frac{\varepsilon}{k} (c_1 P + c_3 G - c_2 \varepsilon) , \]  \hspace{1cm} (6)
is used, where $c_1 = 1.44$, $c_2 = 1.92$, and $c_3 = -0.63$ are model parameters, and $D_\epsilon$ is a transport term similar to (3) as discussed in detail in Umlauf et al. (2003) and Umlauf and Burchard (2005). The form of (6) is motivated by the observation of JHL that a model consisting (1), (4), and (6) could, without any re-calibration, accurately reproduce the turbulent fluxes from their eddy-resolving simulations. Beyond this, it is known from numerous studies that this model has a broad range of applicability including different oceanic entrainment situations, and surface and bottom boundary layer turbulence (Burchard and Baumert 1995; Burchard and Bolding 2001; Umlauf and Burchard 2005). Because of the relative complexity of this model, which will be referred to as the 'full model' in the following, the incorporation into a large-scale ocean climate model was judged to be inefficient by JHL.

Two alternatives with reduced complexity are therefore suggested here, both based on the 'quasi-equilibrium' forms of the stability functions mentioned above. In the first, here referred to as the 'quasi-stationary' model, the rate term in the $\varepsilon$-equation, (6), is neglected. This model has a structure comparable to JHL: both models use (a) essentially identical equations for the TKE balance, (b) diffusion-type equations determining the turbulence length scale, and (c) algebraic expressions describing the effect of stratification on turbulence through a single parameter. However, due to the dependency of (4) on the turbulence quantities $k$ and $\varepsilon$, via the parameter $\bar{N}$, stronger non-linearities compared to JHL are introduced, which may conflict with the general goal of finding a robust and numerically efficient solution with fully implicit methods.

As a second, structurally simpler, alternative it is suggested here to compute the dissi-
pation rate from the equilibrium form of (6),

$$
\varepsilon = \frac{c_1 P + c_3 G}{c_2},
$$

(7)

which is obviously a good approximation in situations where rate and transport terms are small. The turbulent length-scale of this model is constrained by the limiter,

$$
l < c_{\text{lim}} L_b ,
$$

(8)

suggested by Galperin et al. (1988), where $L_b$ is the buoyancy scale defined in (10) below. The computation of a consistent value for the parameter $c_{\text{lim}}$ is discussed by Umlauf and Burchard (2005). Otherwise, there are no differences with respect to the 'quasi-stationary' model. This model will be referred to as the 'algebraic' model.

Since the production terms, $P$ and $G$, are already pre-computed for insertion into (1), the solution of (7) requires only little computational overhead. It is worth noting that, in spite of its simple form, (7) inherits some important key properties from the full equation, (6): models using (6) and (7), respectively, will predict identical entrainment rates in shear-driven situations, since this property is governed by the model behavior in stationary, homogeneous turbulence, where the diffusion terms are negligible by definition (see Burchard and Baumert 1995; Umlauf and Burchard 2005). Note that the same is not true for turbulence near a 'wall', where, similar to the model of JHL, an additional wall function has to be supplied. A wall function is also important for the model’s behavior in unstratified situations; even though easily derived, wall functions will not be discussed here for brevity.
4. The model of Jackson et al.

JHL suggest a TKE balance corresponding to (1) and (3). They argue, however, that in ocean climate models the left hand side of (1) may be ignored since the time scales associated with the evolution of turbulence are much smaller than the numerical advection time scales. They further assume that the turbulent diffusivities appearing in (2) and (3) are identical, i.e. $\nu_k = \nu_t = \nu_\theta$, implying a restriction of their model to weakly stratified flows where the turbulent Prandtl number is close to unity.

The dissipation rate required to close (1) is computed according to

$$\varepsilon = k(c_N N + c_S S),$$

(9)

where $c_N$ and $c_S$ are model constants. As noted by JHL, this model is equivalent to assuming that the dissipative length scale, $l \propto k^{3/2}/\varepsilon$, is a reciprocal average of the buoyancy and shear length scales,

$$\frac{1}{l} \propto \frac{c_N}{L_b} + \frac{c_S}{L_S}; \quad L_b = \frac{k^{3/2}}{N}, \quad L_S = \frac{k^{3/2}}{S}.$$  \hspace{1cm} (10)

Therefore, (9) implies that the dissipative length scale is dominated by the minimum of $L_b$ and $L_S$, which is a simple but physically justifiable modeling assumption (see Schumann and Gerz 1995).

Finally, the turbulent diffusivity in JHL follows from a differential equation that can be written as

$$\nu_\theta = L_b^2 \left( G \frac{\partial^2 \nu_\theta}{\partial z^2} + SR \right),$$

(11)

where $G$ is a constant, and $R$ denotes a monotonically decreasing function of the gradient Richardson number, $Ri$. The functional form of $R(Ri)$ was motivated by the bulk parameter-
ization for entraining gravity currents suggested by Turner (1986), which is known to predict a collapse of turbulence ($R = 0$) for $\tilde{Ri}$ exceeding a prescribed threshold, $Ri_c$. Exactly this model property motivated JHL to include the diffusive term on the right hand side of (11), hence creating the possibility for non-zero mixing due to the 'diffusion of diffusivity' from neighboring regions even if, locally, $\tilde{Ri} > Ri_c$. It should be noted that the appearance (11) and $R(\tilde{Ri})$ entirely follows from ad-hoc assumptions, and involves internal variables that cannot be measured.

5. Model evaluation

a. Model behavior for strong stratification

Interesting insight into the performance of the 'algebraic' model is gained from inserting the production terms defined in (2) and the expression for the diffusivities, (4), into the equilibrium form of the dissipation rate equation, (7). This results in

$$\varepsilon^2 = k^2 \left( \bar{c}_N^2 N^2 + \bar{c}_S^2 S^2 \right),$$

(12)

where

$$\bar{c}_N^2 = -\frac{c_3}{c_2} c_{\mu}^\theta, \quad \bar{c}_S^2 = \frac{c_1}{c_2} c_{\mu},$$

(13)

are functions of $\bar{N}$ as described above. Eq. (12) is recognized as a non-linear relationship between $\bar{N}$ and $\bar{S}$, or, since $\tilde{Ri} = \bar{N}^2/\bar{S}^2$, between $\bar{N}$ and $\tilde{Ri}$. Thus, stability functions of the form $c_{\mu}(\tilde{Ri}), c_{\mu}^\theta(\tilde{Ri})$ can be derived with a pure dependency on the Richardson number. Note that this procedure is completely analogous to the well-known approach of deriving stability functions from the equilibrium assumption, $P + G = \varepsilon$ (see Mellor and Yamada 1974; Canuto
et al. 2001). The important difference is that, here, no equilibrium in the TKE budget is assumed, which is consistent with, and required by, the possibility of diffusive transport of TKE into layers without shear production.

The two forms of the stability functions are compared in Fig. 1, where the focus is on $c_\mu^\theta(Ri)$, controlling the transport of heat, salt, and passive tracers. The traditional form, based on equilibrium in the TKE budget, illustrates some well-known properties like the monotonic decrease from the neutral value towards zero at the critical value, $Ri_c \approx 0.85$, as discussed in great by Canuto et al. (2001). This behavior is contrasted by the form of $c_\mu^\theta(Ri)$ derived from (12), which, although exhibiting a similar monotonic decrease, never falls below the asymptotic value $c_\mu^\theta \approx 0.018$ for $Ri \to \infty$. It is evident that with this model the turbulent fluxes of mass and matter across layers with $Ri \to \infty$ do not vanish, provided a sufficient energy source for the TKE is available.

This interesting behavior can be explained by the observation that (12) reduces to $\varepsilon = \bar{c}_N(Ri)kN$ in the limit of very strong stratification, which, using the definition of the buoyancy scale, (10), and the relation $l \propto k^{3/2}/\varepsilon$, can be re-written as $l = \bar{c}_\text{lim}(Ri)L_b$. This result is particularly revealing since, as easily shown, the new function $\bar{c}_\text{lim}(Ri)$ converges to a finite value for large $Ri$, implying a proportionality between the scale of turbulence, $l$, and the buoyancy scale, $L_b$. This behavior is consistent with turbulence scaling in strongly stratified flows (see Schumann and Gerz 1995), and with the limit for the turbulent length scale, (8).

In this context the following remark is interesting. Experience from laboratory and numerical investigations has consistently shown that stratified turbulence is in full equilibrium only for a single so-called stationary Richardson number, $Ri_{st} \approx 0.15 - 0.25$, if energy
sources other than shear production are negligible (see Shih et al. 2000, and the references therein). This clearly implies that for \( \hat{Ri} > \hat{Ri}_{st} \) stationary turbulence may only be observed if additional energy sources resulting e.g. from transport of turbulence, internal wave effects, and double diffuson contribute to the TKE budget. Some caution is therefore required for the interpretation of stability functions that are based on the equilibrium assumption, \( P + G = \varepsilon \), also for large Richardson numbers as illustrated by Fig. 1. Only for \( \hat{Ri} = \hat{Ri}_{st} \), where \( \hat{Ri}_{st} = 0.25 \) results from the model parameters chosen here, both stability functions coincide, but for larger values of \( \hat{Ri} \) the equilibrium assumption becomes less and less accurate. In particular, the appearance of a critical Richardson number above which turbulence collapses is seen to be an artefact of the equilibrium assumption, rather than an intrinsic property of the second moment closure model as frequently argued.

For the discussion below it is helpful to combine the equilibrium forms of (1) and (6) in order to derive an expression for the mixing efficiency,

\[
\gamma = -\frac{G}{\varepsilon} = \frac{c_2 - c_1}{c_1 - c_3},
\]

with, \( \gamma = 0.23 \) for the parameters used here, i.e. well inside the range of accepted values (Osborn 1980; Shih et al. 2005).

b. Comparison with the model of JHL

For the limiting cases \( N \to 0 \) and \( S \to 0 \), respectively, (12) becomes in form identical to the model of JHL, (9), however, with the difference that the parameters \( \bar{c}_N \) and \( \bar{c}_S \) are functions of \( \hat{Ri} \). Similar to (9), also (12) can be interpreted as an expression relating the dissipative length scale, \( l \propto k^{3/2}/\varepsilon \), to some reciprocal average of the buoyancy and shear
length scales defined in (10), with the respective weighting factors depending on \( Ri \). This is in accordance with a suggestion by Schumann and Gerz (1995), who find that for increasing values of \( Ri \) buoyancy controls the turbulent length scale, rather than shear (see above).

Another interesting similarity may be identified by rewriting (4) and (5) with the help of (10), yielding

\[
\nu_t^0 = L_b^2 S \bar{R}, \quad \bar{R} = c_\mu \bar{N} R_i^{1/2},
\]

(15)

and a similar equation for \( \nu_t \). Comparing (15) with the analogous expression of JHL, (11), reveals a formal similarity only for homogeneous flows since an equivalent of the ad-hoc term describing the 'diffusion of diffusivity' in (11) is missing in (15). The progress made with (15) is, however, that such a term is not at all required: even for \( Ri \rightarrow \infty \), the diffusivity predicted by (15) remains non-zero, provided turbulent diffusion or any other source of TKE is available.

c. A simple test case

In order to investigate the relative performance of the full and equilibrium models, a short comparative model study was performed. Similar to JHL, a turbulent stratified jet was initialized with a Gaussian velocity profile,

\[
u = u_0 \exp \left( - \left( \frac{z}{h_u} \right)^2 \right),
\]

(16)

where \( N^2 = 10^{-6} \text{s}^{-2} \), \( u_0 = 0.2 \text{ ms}^{-1} \), and \( h_u = 10 \text{ m} \) denote the initial values of the buoyancy frequency, speed, and width of the jet chosen here. In contrast to JHL, who forced their simulations to stationarity by nudging towards the initial conditions, here the mean-flow and turbulence parameters were freely evolving in time.
For the one-dimensional (vertical) simulations considered here, the full transport equation for the TKE, (1), with production terms computed according to (2), and the down-gradient transport model, (3), was solved. All other model components of the 'full' and the 'algebraic' models were specified as described above, and all coefficients correspond to those suggested by Umlauf and Burchard (2005) and used by JHL. The evolution of the velocity and temperature fields was computed from the corresponding one-dimensional diffusion equations for momentum and heat. The domain was large enough to exclude any boundary effects, and the numerical resolution and time step corresponded to fully converged runs (except for one run used to test the model performance for very coarse resolution as discussed below).

The temporal evolution of the jet, for brevity not discussed here in all detail, is initiated by a rapid increase of turbulence in two quickly spreading shear layers, leading to symmetric entrainment at the upper and lower flanks, and a supply of energy towards the center. Different to the continuous, self-similar spreading in the unstratified case extensively discussed in the engineering literature (e.g. Pope 2000), here entrainment is interrupted as soon as the shear-production becomes too weak to overcome the damping effect of stratification, and turbulence slowly decays. The computed velocities depicted in Fig. 2 illustrate that, as expected, the largest differences in model performance are observed during the initial phase, where rapid changes in width and vertical structure of the jet occur. The final width and vertical velocity structure predicted by both models are of remarkable similarity, and therefore the following parts of the model intercomparison will be restricted to the late stage where turbulent quantities show a gradual decay. The different durations of the initial stages induce, however, a relative time shift between the two models, thus making it difficult to compare model results at a given time. To overcome this problem, time series of the cross-jet
averaged TKE were computed for both runs, and model results were only compared at the times where the averaged TKE coincided (see Fig. 2).

One of the key parameters predicted by the turbulence models is the turbulent diffusivity, \( \nu_\theta \), displayed in Fig. 3. Both models reproduce the characteristic structure of the diffusivity profile with a local minimum at the center of the jet also found in the eddy-resolving computations of JHL (see below). Diffusivities predicted by the full model are slightly larger, but the overall agreement is surprisingly good given the strongly reduced complexity of the ‘algebraic’ model. To check the robustness of the model for coarse-resolution climate modeling applications, results were re-computed with a resolution of 10 m, i.e. resolving the jet with a few grid points only. Fig. 3 illustrates that the computed diffusivity is only marginally different from that of the fully converged run (again, the relative time shift was corrected as described above).

The physical processes generating the diffusivity profiles described above are illustrated in Fig. 4. In a qualitatively similar way, both models suggest that turbulence is produced by shear production in the flanks of the jet, and is transported by turbulent diffusion towards the center and the boundaries. In the center of the jet, the transported TKE is used to generate a buoyancy flux, \( G \), even in the absence of shear production. The scaled buoyancy production shown in Fig. 4 corresponds to the well-known mixing efficiency, \( \gamma \), with values varying around \( \gamma \approx 0.2 \), i.e. close to the equilibrium value predicted by (14), and in agreement with commonly observed values (Osborn 1980; Shih et al. 2005). Quantitatively, different predictions between both models are observable mainly in the profiles for the scaled buoyancy production and the rate term. In summary, also the TKE budgets predicted by both models are of acceptable similarity, indicating that the ‘algebraic’ model is a simple, but useful,
alternative to the full model even in this relatively complex flow.

d. Comparison with eddy-resolving simulations

As an independent benchmark, model results were compared to the eddy-resolving simulations of the stratified jet described in detail by JHL. Briefly, JHL obtained reference profiles for turbulence quantities by directly solving the Navier-Stokes equations with a relaxation in order to obtain a stationary velocity profile, similar to (16), and a stationary stratification profile. The vertical shear and stratification, $S$ and $N$, from these eddy-resolving simulations were used to drive the second-moment models in the following analysis. The stationary velocity profile and the corresponding diffusivity from JHL are displayed in Fig. 5, along with the diffusivities computed from the 'quasi-stationary' and 'algebraic' models, respectively. The 'quasi-stationary' model is in excellent agreement with the results of JHL, which is impressive since model parameters have been calibrated in no way for this flow. The 'algebraic' model, also shown in Fig. 5 qualitatively reproduces the structure of the diffusivity profile, but underestimates the observed mixing. Also the TKE budgets from the simulations of JHL and the 'quasi-stationary' model, illustrated in Fig. 6, exhibit a remarkable similarity. The dominant physical mechanisms identified here are again the production of turbulence by the strong shear in the flanks of the jet, and the transport of TKE towards the center and the edges where it is available for mixing.
6. Discussion and conclusions

In large-scale ocean models relying on the stability function approach, a collapse of mixing of heat, salt, and matter across layers with $Ri \to \infty$ is observed even if a sufficient energy supply for mixing would be available. This blocking of transport across layers without shear becomes a problem especially in isopycnal ocean models with very low artificial (numerical) mixing that may otherwise, at least partly, mask this undesirable model property.

It has been shown here that, in the context of second-moment modeling, this behavior is a simple consequence of the equilibrium assumption, $P + G = \epsilon$, that is not valid for $Ri \gg Ri_{st}$, where $Ri_{st} \approx 0.25$ denotes the Richardson number for stationary homogenous turbulence in equilibrium. If, as in the case investigated here, the vertical diffusion of TKE is the major energy source in layers without shear production this problem may be overcome with two obvious changes in the model structure: (a) a model for the turbulent transport of TKE has to be included, and, (b), the equilibrium assumption has to be discarded when deriving expressions for the stability functions. Interestingly, the latter condition was shown to immediately lead to stability functions without critical Richardson number thus revealing the existence of a finite $Ri_c$ as an artefact of the equilibrium assumption. This conclusion is consistent with the simple argument that mixing may occur at any Richardson number if a sufficient energy supply other than shear production is available.

It is interesting to compare these findings with a recent suggestion by Canuto et al. (2008), who introduced stability functions without critical Richardson number by modifying the relaxation time scales for the second moments, which is a rather different approach. Assuming equilibrium in the TKE budget, Canuto et al. (2008) have shown that this results
in stability functions predicting finite mixing of momentum for all values of $Ri$; however, the predicted mixing of heat, salt, and passive tracers vanishes for $Ri \rightarrow \infty$. This behavior is consistent with the dominance of internal waves at large values of $Ri$ but clearly not with the data investigated here, where diffusion of turbulence energizes the mixing across layers with $Ri \rightarrow \infty$. Since the present approach focuses on the consequences and limitations of the equilibrium assumption, leaving the modeling of the second-moment equations untouched, no contradiction arises here, and both approaches are in fact easily combined.

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REFERENCES


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