Assignment Nr. 3

due 25 May

Problem 1

Starting from the momentum budget and material law for a Newtonian Boussinesq fluid in (2.1) and (2.3), respectively, in the lecture notes, show that the evolution equation for the *kinetic energy* can be expressed as:

$$\frac{\mathrm{D}E}{\mathrm{D}t} = -\frac{1}{\rho_0}\frac{\partial u_i p}{\partial x_i} + \frac{\partial}{\partial x_i}\Big(2\nu u_j S_{ij}\Big) - 2\nu S_{ij}S_{ij} - \frac{\rho g u_3}{\rho_0} \,, \tag{1}$$

where u_i denotes the components of the velocity vector, S_{ij} those of the rate of deformation tensor, p the pressure, g the acceleration of gravity, and ρ the density. The diffusivity of momentum is referred to as ν, and the constant reference density as ρ_0. Show, term by term, that this equation is equivalent to (2.16) in the lecture notes.

Problem 2

Consider Eq. (2.26) in the lecture notes, which describes the evolution of the vorticity, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, for a Boussinesq fluid with constant diffusivity, ν.

- Show that the *enstrophy*, $\omega^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$, evolves according to

$$\frac{\mathrm{D}\omega^2}{\mathrm{D}t} = 2\omega_i\omega_j S_{ij} + 2\epsilon_{ijk}\omega_i \frac{\partial \rho}{\partial x_j}\frac{g_k}{\rho_0} + \nu \frac{\partial^2\,\omega^2}{\partial x_k \partial x_k} - 2\nu \frac{\partial \omega_i}{\partial x_j}\frac{\partial \omega_i}{\partial x_j} \,, \tag{2}$$

  where S_{ij} denots the components of the rate of strain tensor, and g_i the components of the acceleration of gravity.

- What can be said about the sign of the last term in (2)? Is this term always a sink of enstrophy?
- Discuss the physical meaning of every term appearing in (2).
- Show that for a two-dimensional (plane) flow of an inviscid fluid with $\rho = \rho_0$, the vorticity equation in (2.26) reduces to

$$\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0 \,, \tag{3}$$

  where ω denotes the magnitude of the vorticity component perpendicular to the plane of the flow. Does vortex-stretching occur in such flows?

Problem 3

Recalling that the evolution of an infinitesimal material line element, $\mathrm{d}\boldsymbol{x}$, is described by

$$\frac{\mathrm{D}(\mathrm{d}x_i)}{\mathrm{D}t} = \frac{\partial u_i}{\partial x_j}\mathrm{d}x_j \; , \tag{4}$$

derive and equation for the evolution of $\mathrm{d}s^2 = \mathrm{d}\boldsymbol{x} \cdot \mathrm{d}\boldsymbol{x}$, where $\mathrm{d}s$ denotes the length of the element. Discuss the relation to the enstrophy equation (2) for an inviscid fluid with constant density.