Assignment Nr. 6

due 06 July

Problem 1

This problem uses again the velocity time series from a turbulent bottom boundary layer in the Western Baltic Sea already discussed in Assignment 4. Again, you need to rotate the horizontal velocity components, such that u denotes the component pointing into the direction of the mean flow, v the spanwise component, and w the vertical component. As done in Assignment 4, please remove the mean from all three components and only work with the fluctuating parts in the following. Recall that the sampling frequency was $f = 25$ Hz.

Using a Discrete Fourier Transform (DFT), numerical software packages are able to compute a discrete approximation $E_m(\omega_m)$ of the energy spectrum $E(\omega)$ of a discrete time series of length $T = N\Delta T$, where $\omega_m = 2\pi m/T$ are the discrete frequencies, and ΔT is the sampling interval ($m = 0, 1, 2, \ldots, N$). In MATLAB, one possibility to compute E_m is the use of the function `pwelch`, where parameters should be adjusted such that sub-spectra are computed over half-overlapping Hamming windows of (approximately) length $N/10$. A similar functionality exists also in most other software packages.

(a) Compute the one-dimensional energy spectra E_m for the velocity time series of u, v, and w, respectively, and plot them in a single double-logarithmic (!) plot as a function of ω_m.

(b) Using Taylor's assumption of "frozen turbulence", how would you convert $E_m(\omega_m)$ into a spatial wavenumber spectrum $\tilde{E}_m(K_m)$, based on the known mean horizontal advection speed U? You don't need to actually do this transformation — just explain the steps involved. Hint: the transformation must have the property $\int E(\omega)\mathrm{d}\omega = \int \tilde{E}(K)\mathrm{d}K$ because both integrals represent the same total variance ($\langle u'^2\rangle$, $\langle v'^2\rangle$, or $\langle w'^2\rangle$, depending on which velocity component is considered).

(c) At low frequencies, which velocity component contains the least energy? Can you explain this? Hint: recall that the measurements are taken close to the bottom.

(d) Plot the vertical velocity spectrum together with a curve with slope $\omega_m^{-5/3}$, as expected for the inertial subrange. For which frequency range do the slopes of the two spectra approximately coincide?

(e) Assume that in the inertial subrange the spectrum is represented by

$$E_m = C\varepsilon^{2/3}\omega_m^{-5/3} \,, \tag{1}$$

where C is a constant, and ε denotes the dissipation rate. Describe a method to compute the dissipation rate from the spectra you computed above.

Problem 2

Is has been shown in the lecture that, for isotropic turbulence with velocity variance $\langle u'^2 \rangle$, the correlation tensor $\boldsymbol{R}(\boldsymbol{r}, t)$ can be expressed as

$$R_{ij}(\boldsymbol{r}, t) = \langle u'^2 \rangle \left(g\delta_{ij} + (f - g)\, \frac{r_i r_j}{r^2} \right) \ , \tag{2}$$

where $\boldsymbol{r}$ is a vector of length r connecting the two points for which the correlation is computed, t is time, and $f(r, t)$ and $g(r, t)$ are the longitudinal and transverse correrlation functions, respectively. Assume for all following derivations that turbulence is isotropic.

(a) Starting from the general definition, $R_{ij} = \langle u'_i(\boldsymbol{x}) u'_j(\boldsymbol{x} + \boldsymbol{r}) \rangle$, show that for an incompressible fluid

$$\frac{\partial}{\partial r_j} R_{ij} = 0 \quad . \tag{3}$$

(b) Show that for any vector $\boldsymbol{r}$ with length r the following relations hold:

$$\frac{\partial r}{\partial r_i} = \frac{\partial (r_k r_k)^{1/2}}{\partial r_i} = \frac{r_i}{r} \ , \qquad \frac{\partial}{\partial r_j} \left(\frac{r_i r_j}{r^2} \right) = \frac{2 r_i}{r^2} \quad . \tag{4}$$

(c) Using the expression in (2), and the results derived above, show that for incompressible flows the following relation holds for the longitudinal and transverse correlation functions:

$$g = f + \frac{1}{2} r \frac{\partial f}{\partial r} \quad . \tag{5}$$

So you just showed that for isotropic turbulence, the correlation functions f and g are not independent. This implies that the correlation tensor in (2) is determined by a single scalar function of r.

Problem 3

For isotropic turbulence, the components of the velocity spectrum tensor can be written in the following form:

$$\Phi_{ij}(\boldsymbol{K}, t) = A(K, t)\delta_{ij} + B(K, t) K_i K_j \ , \tag{6}$$

where $K = |\boldsymbol{K}|$ is the magnitude of the wavenumber vector $\boldsymbol{K}$.

(a) Show that for incompressible turbulence the following relation holds:

$$K_j \Phi_{ij}(\boldsymbol{K}, t) = 0 \quad . \tag{7}$$

Hint: Use relation (3) derived above, and start from the general definition of the correlation tensor R_{ij} as the Fourier transform of Φ_{ij} given in the lecture notes.

(b) Use (6) and (7) to show that for incompressible, isotropic turbulence the spectrum tensor is of the form

$$\Phi_{ij}(\boldsymbol{K}, t) = A(K, t) \left(\delta_{ij} - \frac{K_i K_j}{K^2} \right) \quad . \tag{8}$$